GRAPHIC ALGEBRA

OR

GEOMETRICAL INTERPRETATION

OF THE

THEORY OF EQUATIONS

OF

ONE UNKNOWN QUANTITY

BY

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PREFACE.

Of late years Graphical Methods have been used with great profit in many departments of science. Such methods give a clear and comprehensive view of the facts of observation, and, by suggesting new relations, furnish a valuable means of formulating new laws. Their importance in solving many problems in applied mathematics is also very generally recognized. In interpreting analytical expressions and illustrating analytical processes in pure mathematics such methods are equally valuable. They accustom the student to take into account not merely one detail at a time of a complex operation, but show at a glance all the results in their proper relations.

Such methods are well adapted to the presentation of the principles and processes of the Theory of Equations. The Theory of Equations is a subject so essential in all the applications of Mathematical Analysis that the student should be made familiar with it early in his course. But at this stage he is not prepared to appreciate the very general analytical demonstrations found in treatises on this subject. He is not accustomed to such methods of reasoning, and, although he may not be able to find any flaws in the logic, he is not familiar with the conceptions employed, and cannot give any tangible form to his ideas. Therefore they are vague and unsatisfactory. Knowledge so sought is soon lost, and the only permanent result of such study is a loose way of thinking. On the other hand, the Graphical Method presents

these abstract principles in a concrete form. Each principle therefore becomes firmly impressed on the mind, and the student escapes the tendency to substitute a form of words for clear conceptions of ideas.

This work is based on some years' experience in teaching the Theory of Equations to the Freshman Class in Yale College, and is adapted to students who have mastered quadratic equations and the principle of similar triangles.

Its object is to give a clear notion of the Geometrical meaning of the Theorems and operations of the Theory of Equations in their application to the solution of numerical equations of one unknown quantity, and also to furnish a suitable introduction to Analytical Geometry and the Calculus.

The development of the subject from the graphical standpoint is thought to be very complete. The difficulties are presented singly, and the student is led up to general propositions by means of simple illustrative examples.

Some practical subjects usually treated in the Theory of Equations have been omitted, such as Newton's Method of Approximation and Sturm's Theorem.

The principle involved in Newton's Method is the same as that in Horner's Method, but the latter can be applied to the class of equations here treated in a more elegant and systematic manner.

The use of Sturm's Theorem may be avoided by plotting the equation, since the plot usually shows whether the roots are real or imaginary, and in critical cases Horner's Method may be used in connection with the plotting as effectively as the method of Sturm, and with less labor.

New Haven, September 5, 1882.

CONTENTS.

CHAPTER I.

CHAPTER II.

CHAPTER III.

CHAPTER IV.

CHAPTER V.

GRAPHIC ALGEBRA.

CHAPTER I.

Plotting Equations.

Art. 1. Every Equation containing two unknown quantities may be represented by a *line*, either straight or curved. The method of drawing this line is similar to that employed in making an outline map. For example, to draw a map of the coast of South America, it is best to find the Latitude and Longitude of a number of places, the latitudes being measured North and South from the Equator, and the longitudes East and West from any convenient Meridian. North latitudes are called + and South latitudes −, East longitudes + and West longitudes −.

2. The following table contains the latitudes and longitudes of places on the coast of South America. The places are selected at intervals of two degrees of longitude. In drawing the map, it will be convenient to use paper ruled in small checks. One of the lines may represent the Equator and one at right angles to it the Meridian of reference. East longitudes are measured on the Equator to the right of

this meridian, and West longitudes to the left. North latitudes are measured upward from the Equator and South latitudes downward, each division of a line representing two degrees. The intersection of the Equator and the Meridian of reference is called the *Origin.*

TABLE.

Longitude.	Latitude.	Longitude.	Latitude.
0°	+ 9°.0	− 2°	+ 10°.1
+ 2	+ 6.8	− 4	+ 10.7
+ 4	+ 6.2	− 6	+ 10.4
+ 6	+ 6.0	− 8	+ 10.6
+ 8	+ 4.9	− 10	+ 12.0
+ 10	+ 2.1	− 12	+ 12.4
+ 12	0.	− 14	+ 11.7
+ 14	− 1.3	− 16	+ 10.0
+ 16	− 2.0	− 18	+ 8.3
+ 18	− 2.3	− 20	+ 0.4
+ 20	− 2.5	− 22	− 3.9
+ 22	− 3.8		
+ 24	− 5.0		
+ 26	− 8.0		

The longitude of the first place is 0° and its latitude + 9°. Its place on the map therefore is on the meridian four and a half spaces above the origin. To fix the second point on the map, measure two degrees to the right of the origin on the

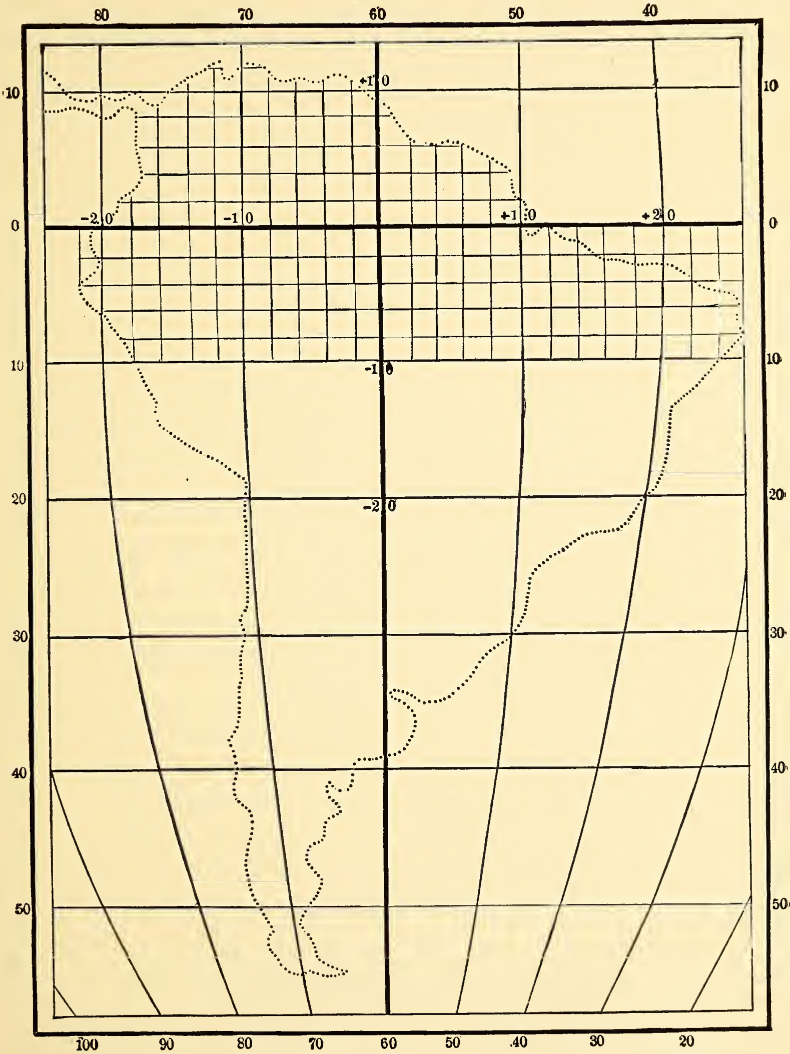
80
70
60
50
40
10
0
10
20
30
40
50
+1 0
−2 0
−1 0
+1 0
+2 0
−1 0
−2 0
100
90
80
70
60
50
40
30
20

FIG. 1.

Equator and 6°.8 upward on the meridian passing through this 2° point. In the same way fix the position of the other places from the values in the table. Draw a line through all the points thus laid down. This will be an outline of the coast as far as measured. The outline may be drawn as accurately as desired, by taking the latitudes and longitudes of points at sufficiently small intervals.

3. In the same way an outline can be drawn of a river, a road, a boundary line, etc. Instead of the Equator and a Meridian, any convenient lines of reference may be selected, and each division on the paper may represent any convenient unit of measure, as the mile, rod, chain, etc.

Examples.

1. Draw a map of the Ohio River from the following latitudes and longitudes, which are reckoned from the Equator and the meridian of Washington respectively:

Long. W.	Lat. N.	Long. W.	Lat. N.	Long. W.	Lat. N.
0°	+ 45°.0	− 5°	+ 39°.0	− 10°	+ 38°.0
− 1	+ 44.0	− 6	+ 38.4	− 11	+ 37.9
− 2	+ 41.5	− 7	+ 38.6	− 12	+ 37.2
− 3	+ 40.3	− 8	+ 39.0	− 13	+ 35.4
− 4	+ 40.5	− 9	+ 38.3	− 14	+ 33.0

2. Draw the map of a river from the following measurements (Fig. 2):

Miles E. & W.	Miles N. & S.	Miles E. & W.	Miles N. & S.	Miles E. & W.	Miles N. & S.
0	+ 3.6	+ 6	− 4.1	− 4	− 3.6
+ 1	+ 3.9	+ 7	− 4.7	− 5	− 4.2
+ 2	+ 5.1	+ 8	− 4.0	− 6	− 3.7
+ 3	+ 7.0	− 1	+ 3.6	− 7	− 3.1
+ 4	+ 5.8	− 2	+ 2.4	− 8	− 5.2
+ 5	− 1.0	− 3	− 0.2	− 9	− 5.0

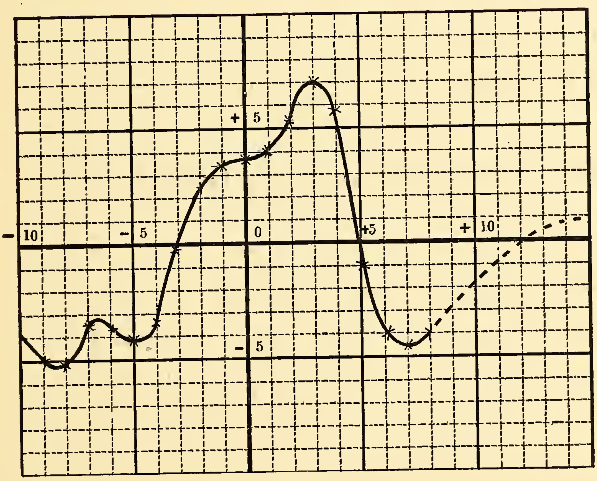

FIG. 2.

3. Draw a line through points located as follows:

Inches.	Inches.	Inches.	Inches.	Inches.	Inches.
0	+ 3.4	+ 7	+ 4.6	+ 14	− 1.0
+ 1	+ 4.0	+ 8	+ 4.4	+ 15	− 5.0
+ 2	+ 4.4	+ 9	+ 4.0	− 1	+ 2.8
+ 3	+ 4.6	+ 10	+ 3.4	− 2	+ 1.9
+ 4	+ 4.7	+ 11	+ 2.8	− 3	+ 0.8
+ 5	+ 4.8	+ 12	+ 1.9	− 4	− 1.0
+ 6	+ 4.7	+ 13	+ 0.8	− 5	− 5.0

4. Draw the plot of a river from the following measurements:

Miles.	Miles.	Miles.	Miles.	Miles.	Miles.
0	− 6.	+ 7	+ 6.	− 4	+ 3.
+ 1	− 4.5	+ 8	+ 7.	− 5	+ 3.5
+ 2	− 3.	+ 9	+ 6.5	− 6	+ 3.3
+ 3	− 1.	+ 10	+ 5.	− 7	+ 3.
+ 4	+ 0.5	− 1	− 5.	− 8	+ 2.
+ 5	+ 2.	− 2	− 4.	− 9	+ 1.
+ 6	+ 4.	− 3	0.	− 10	0.

5. Plot a line from the following table of values:

0	+ 1.	+ 4	+ 3.	− 1	+ 0.5
+ 1	+ 1.5	+ 5	+ 3.5	− 2	0.
+ 2	+ 2.	+ 6	+ 4.	− 3	− 0.5
+ 3	+ 2.5	+ 7	+ 4.5	− 4	− 1.

6. Plot the following line:

0	0.	+ 8	− 0.87	− 4	− 0.87
+ 1	+ 0.50	+ 9	− 1.	− 5	− 0.50
+ 2	+ 0.87	+ 10	− 0.87	− 6	0.
+ 3	+ 1.	+ 11	− 0.50	− 7	+ 0.50
+ 4	+ 0.87	+ 12	0.	− 8	+ 0.87
+ 5	+ 0.5	− 1	− 0.50	− 9	+ 1.
+ 6	0.	− 2	− 0.87	− 10	+ 0.87
+ 7	− 0.5	− 3	− 1.	− 11	+ 0.50

7. Plot a line from the following table of values:

0	− 3.	− 1	− 2.	− 5	+ 2.
+ 1	− 4.	− 2	− 1.	− 6	+ 3.
+ 2	− 5.	− 3	0.	− 7	+ 4.
+ 3	− 6.	− 4	+ 1.	− 8	+ 5.

8. Plot the following line:

0.	− ∞	+ 1.4	+ 0.15	+ 2.8	+ 0.45
+ 0.2	− 0.70	+ 1.6	+ 0.20	+ 3.	+ 0.48
+ 0.4	− 0.40	+ 1.8	+ 0.25	+ 3.2	+ 0.50
+ 0.6	− 0.22	+ 2.0	+ 0.30	+ 3.4	+ 0.53
+ 0.8	− 0.10	+ 2.2	+ 0.34	+ 3.6	+ 0.55
+ 1.	0.	+ 2.4	+ 0.38	+ 3.8	+ 0.57
+ 1.2	+ 0.08	+ 2.6	+ 0.42	+ 4.	+ 0.58

4. By a process similar to the preceding, a *line* may be drawn to represent an *equation*. This is called plotting.

The two lines of reference will be called *axes;* the one extending right and left the *x-axis*, the other the *y-axis*. Their intersection is called the *origin*. The distances to the *right* of the origin are *plus*, and to the *left minus*. Distances reckoned *upward* from the *x-axis* are *plus*, and *downward minus*.

Take as an example the equation

$$y = x^3 - 5x^2 + 2x + 6.$$

To plot this, x is not to be considered an unknown quantity with a fixed value as is customary in solving equations, but a quantity which may have any value whatever. It is then called a *variable*, and is conceived to assume in succession all possible values, positive and negative. Each value of x substituted in the above equation will give a corresponding value of y; then y is also a *variable*, and each value of x with the corresponding value of y will locate a point. The line connecting the points so determined forms the *plot*.

If in the above equation we assume $x = 0$, then $y = 6$. When $x = 1$, $y = 4$, etc. In this way the following table of values is formed.

x	y
0	+ 6
+ 1	+ 4
+ 2	− 2
+ 3	− 6
+ 4	− 2
+ 5	+ 16
− 1	− 2
− 2	− 26

Having drawn the x and y *axes*, begin at the origin, where $x = 0$, and measure upward 6 units on the *y-axis*, because $y = +6$, and mark the point so determined. To fix the second point, we measure one unit to the right, because $x = +1$, and thence measure upward 4 units, because $y = +4$. A third point will be situated 2 units to the right of the *y-axis* and 2 units below the *x-axis*. Thus we may locate all the points in the table of values. The line drawn through these *points taken in order* will be the plot of the equation. (Fig. 3.)

In the same way any equation of the form

$$y = x^n + Ax^{n-1} + \ldots + Px + Q$$

may be plotted.

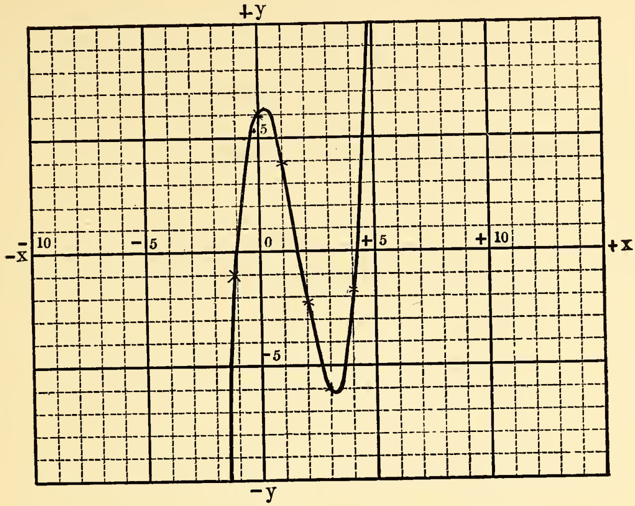

FIG. 3.

5. It is important to join the points *in order*. To effect this, begin at the extreme right-hand point, and draw the line through the next point towards the left, and so on through the whole series of points.

6. It is necessary to assume only so many values of x positive and negative as will show the ultimate directions of the curve, and to choose such intervals between successive values as will fix the turning points of the curve, and the places where it crosses the axes. In order to determine with sufficient accuracy how the line curves, it is often necessary to assume values of x at intervals of less than a unit.

7. It is often convenient and, as will be seen hereafter, legitimate to use one scale for plotting values of x and a different one for values of y.

Examples.

1. Plot the equation $y = x^2 - 2x - 3$.

x	y
0	-3
$+1$	-4
$+2$	-3
$+3$	0
$+4$	$+5$
$+5$	$+12$
-1	0
-2	$+5$
-3	$+12$

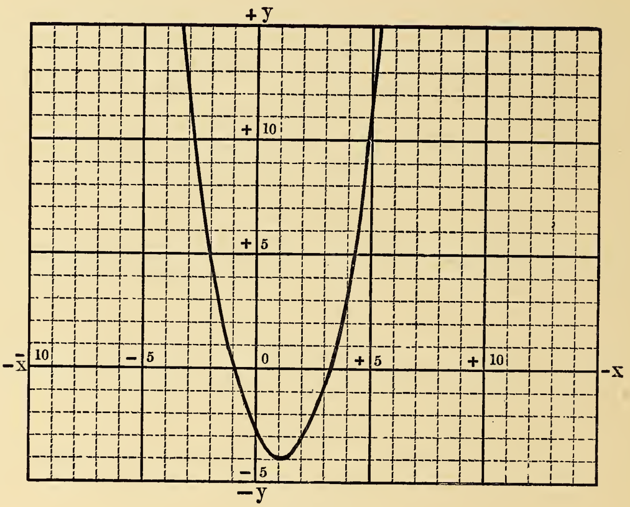

FIG. 4.

8. It is evident from this equation that if the values of x increase beyond those given in the table, each value of y will be larger numerically than the preceding; therefore the table of values so far as given is sufficient to determine the ultimate directions of the curve. In general, when this is found to be the case, it is not necessary to compute any more values for the table.

2. Plot the equation $y = x^3 - 6x^2 + 11x - 2$.

The values of x 1, 2, and 3 give points of the curve at the

same distance above the *x-axis*; but if we assume intermediate values of x, it will be found that values of x between 1 and 2 give values of y greater than 4, and values of x between 2 and 3 give values of y less than 4. In general,

x	y
0	-2
$+1$	$+4$
$+2$	$+4$
$+3$	$+4$
$+4$	$+10$
$+5$	$+28$
-1	-20
-2	-56
$+0.5$	$+2.1$
$+1.5$	$+4.4$
$+2.5$	$+3.6$
$+3.5$	$+5.9$

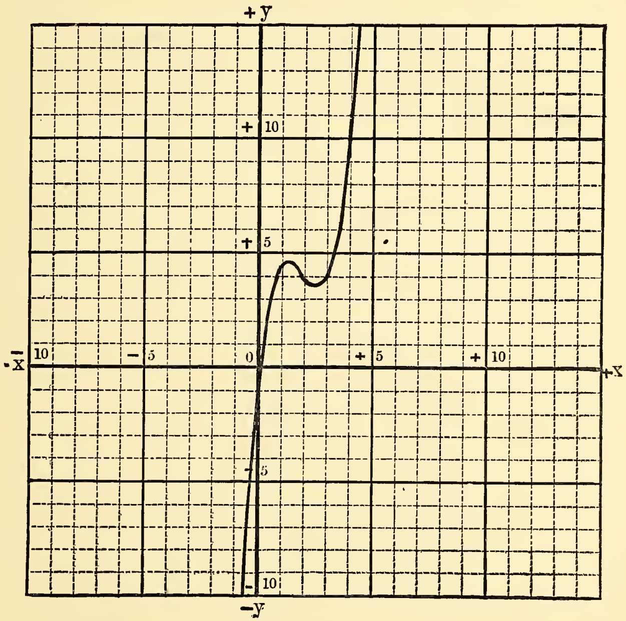

FIG. 5.

when any doubt exists as to the form of the curve between any two consecutive points, the form can be determined by assuming intermediate values.

3. Plot the equation $y = x^2 - 4x + 8$.

4. Plot the equation $y = x^3 + 3x^2 + 2x$.

5. Plot the equation $y = x^4 + x^3 - 13x^2 - x + 12$.

x	y
0	+ 12
+ 1	0
+ 2	− 18
+ 3	0
+ 4	+120
− 1	0
− 2	− 30
− 3	− 48
− 4	0
− 5	+192

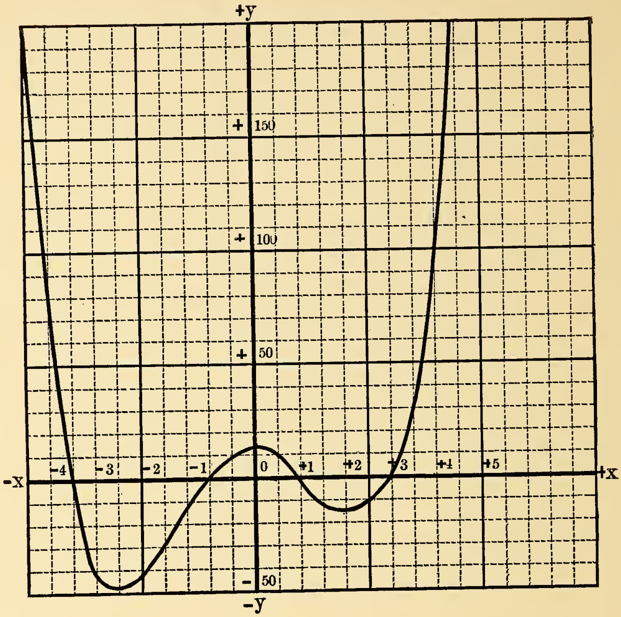

FIG. 6.

In this example the values of y have been plotted on a smaller scale than the values of x, in order to bring the plot within more convenient limits. For values of x two divisions on the paper represent one unit, and for values of y one division represents ten units. This does not change the position of the points where the curve crosses the *x-axis*, neither does it change the relative positions of the highest and lowest points on the curve

6. Plot the equation $y = x^4 - x^3 - 27x^2 + 25x + 5$
7. Plot the equation $y = x^3 + 11x^2 + 36x + 15$.
8. Plot the equation $y = -3x + 2$.

A Short Method of Substitution.

9. In the equation $y = 2x^4 - 6x^3 + 10x^2 - 15x + 10$, to find the value of y when $x = 2$ proceed as follows:

$$\begin{array}{rrrrrl} 2x^4 & -\ 6x^3 & +\ 10x^2 & -\ 15x & +\ 10 & \underline{|\ x = 2} \\ & +\ 4x^3 & -\ 4x^2 & +\ 12x & -\ 6 & \\ \hline & -\ 2x^3 & +\ 6x^2 & -\ 3x & +\ 4 & \text{Whence } y = 4. \end{array}$$

When $x = 2$, $2x^4 = 2 . x . x^3 = 4x^3$. Adding this to $-6x^3$ gives $-2x^3$ for the sum of the first two terms.

Again, when $x = 2$, $-2x^3 = -4x^2$. Adding this to the third term $10x^2$, gives $+6x^2$, which is the sum of the first three terms.

This, again, when $x = 2$ becomes $12x$, which added to $-15x$ makes $-3x$ for the value of the first four terms.

This in turn is equal to -6, which added to the $+10$ gives $+4$ for the value of y when $x = 2$.

In practice it is only necessary to set down the coefficients thus:

$$\begin{array}{rrrrrl} 2 & -\ 6 & +\ 10 & -\ 15 & +\ 10 & \underline{|\ 2} \\ & +\ 4 & -\ 4 & +\ 12 & -\ 6 & \\ \hline & -\ 2 & +\ 6 & -\ 3 & +\ 4 & \text{Whence } y = 4 \end{array}$$

When $x = 3$,

$$\begin{array}{l} 2 - 6 + 10 - 15 + 10 \,\underline{|\;3\;} \\ \quad\;\; 6 \qquad 0 + 30 + 45 \\ \hline \quad\;\; 0 + 10 + 15 + 55 \end{array} \quad \text{Whence } y = 55$$

When $x = -1$,

$$\begin{array}{l} 2 - 6 + 10 - 15 + 10 \,\underline{|\;-1\;} \\ \;\; - 2 + \;8 - 18 + 33 \\ \hline \;\; - 8 + 18 - 33 + 43 \end{array} \quad \text{Whence } y = 43$$

By this process a table of values for plotting the equation can be very rapidly calculated.

10. If any one of the successive terms in an equation is wanting, its place should be represented by 0. Thus the equation

$$y = x^4 - 3x^3 + 4x + 5,$$

where x^2 is wanting, may be written

$$y = x^4 - 3x^3 + 0x^2 + 4x + 5;$$

and if $x = 2$, we have

$$\begin{array}{l} 1 - 3 + 0 + 4 + 5 \,\underline{|\;2\;} \\ \quad + 2 - 2 - 4 \quad 0 \\ \hline \quad - 1 - 2 \quad 0 + 5 = y \end{array}$$

RULE. *Write the coefficients of the successive terms of the second member of the equation in the order of the powers of x, beginning with the coefficient of the highest power, prefixing*

the proper signs, and supplying the place of any omitted term by o.

Multiply the first coefficient by the number to be substituted for x, and add the product to the second coefficient. Multiply this sum by the number to be substituted for x, and add the product to the third coefficient. Thus proceed to the last term. The last sum will be the value of y.

11. The following example shows how this method may be used to form a table of values for plotting an equation:

$$y = x^4 - 2x^3 - x^2 + 2x.$$

Supplying the place of the absolute term by o, the coefficients may be written

$$1 - 2 - 1 + 2 + 0.$$

When $x = 0$ it is evident that $y = 0$.

x	y
0	0
$+1$	0
$+2$	0
$+3$	$+24$
-1	0
-2	$+24$
$+.5$	$+.56+$
$+1.5$	$-.94^-$
$-.5$	$-.94^-$

When $x = 1$,

$$\begin{array}{rrrrr|l} 1 & -2 & -1 & +2 & +0 & \underline{1} \\ & 1 & -1 & -2 & 0 & \\ \hline & -1 & -2 & 0 & 0 = y & \end{array}$$

When $x = 2$,

$$\begin{array}{rrrrr|l} 1 & -2 & -1 & +2 & +0 & \underline{2} \\ & 2 & 0 & -2 & +0 & \\ \hline & 0 & -1 & 0 & 0 = y & \end{array}$$

When $x = 3$,

$$\begin{array}{rrrrr|l} 1 & -2 & -1 & +2 & +0 & \underline{3} \\ & 3 & +3 & +6 & +24 & \\ \hline & +1 & +2 & +8 & +24 = y & \end{array}$$

When $x = -1$,

$$\begin{array}{rrrrr|l} 1 & -2 & -1 & +2 & +0 & \underline{-1} \\ & -1 & +3 & -2 & 0 & \\ \hline & -3 & +2 & 0 & 0 = y & \end{array}$$

When $x = -2$,

$$\begin{array}{rrrrr|l} 1 & -2 & -1 & +2 & +0 & \underline{-2} \\ & -2 & +8 & -14 & +24 & \\ \hline & -4 & +7 & -12 & +24 = y & \end{array}$$

When $x = +.5$,

$$\begin{array}{rrrrr|l} 1 & -2 & -1 & +2 & +0 & \underline{.5} \\ & .5 & -.75 & -.875 & +.5625 & \\ \hline & -1.5 & -1.75 & +1.125 & +.5625 = y & \end{array}$$

When $x = +1.5$,

$$\begin{array}{rrrrr|l} 1 & -2 & -1 & +2 & +0 & \underline{1.5} \\ & +1.5 & -.75 & -2.625 & -.9375 & \\ \hline & -.5 & -1.75 & -.625 & -.9375 = y & \end{array}$$

When $x = -.5$,

$$\begin{array}{rrrrr|l} 1 & -2 & -1 & +2 & +0 & \underline{-.5} \\ & -.5 & +1.25 & -1.25 & -.9375 & \\ \hline & -2.5 & +.25 & +1.875 & -.9375 = y & \end{array}$$

CHAPTER II.

Formation of Equations. Roots of Equation. Double Position.

Definitions.

12. An equation of one unknown quantity is of the nth degree when the highest exponent of the unknown quantity is n. For example, equations of the second, third, and fourth degrees are those in which the highest exponents of x are 2, 3, and 4 respectively.

13. An equation of the second degree is called a *quadratic*, of the third degree a *cubic*, and of the fourth degree a *biquadratic*.

14. It is convenient in discussing equations to transpose all terms to the first member, and arrange them in the order of the powers of x. This form,

$$x^n + Ax^{n-1} + Bx^{n-2} \dots\dots Px + Q = 0,$$

is called the *general equation of the nth degree.*

15. A *root* of an equation is a number which when substituted for x satisfies the equation; that is, makes the two members equal. Therefore, in the general form of the equation, the root reduces the first member to zero.

16. Any or all the roots of an equation may be *imaginary.*

17. In treating equations of one unknown quantity by the Graphical Method, we transpose all terms to the first member and put y equal to this member. For example, in the equation

$$x^4 - 4x^3 + x^2 + 7x = 3;$$

transposing,

$$x^4 - 4x^3 + x^2 + 7x - 3 = 0.$$

Writing y equal to the first member,

$$y = x^4 - 4x^3 + x^2 + 7x - 3.$$

This equation is of the form given in the last chapter, and may be plotted by the method therein explained. The plot is as follows:

x	y
0	-3
1	$+2$
2	-1
3	0
4	$+41$
-1	-4
-2	$+35$

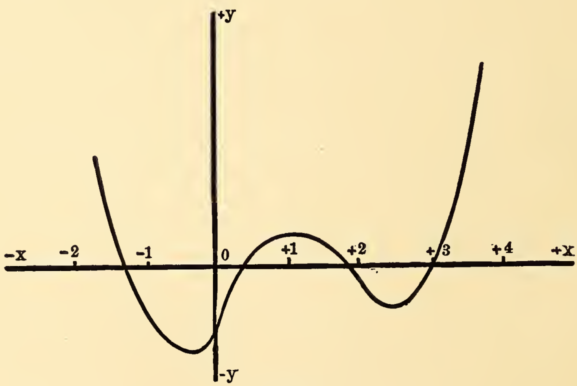

FIG. 7.

18. The ROOTS of the equation

$$x^4 - 4x^3 + x^2 + 7x - 3 = 0$$

are the distances measured from the origin along the *x-axis* to the points where the curve intersects this axis. For these

distances represent values of x which reduce the first member to zero.

19. *All the real roots of an equation may be found exactly or approximately by plotting it.*

Formation of Equations.

20. If we solve the equation

$$x^2 - 5x + 6 = 0$$

by the ordinary rule for quadratics, we obtain

$$x = 2 \qquad \text{and} \qquad x = 3;$$

whence

$$x - 2 = 0 \quad \text{and} \qquad x - 3 = 0.$$

The product of these last two equations, member by member, will be the original equation; that is,

$$(x - 2)(x - 3) = x^2 - 5x + 6 = 0.$$

It might be shown that the first member of any quadratic equation can be resolved into two factors, $x - a$ and $x - b$, when a and b are the roots. We may thus form a quadratic equation, when its roots are given, by multiplying together the expressions formed by subtracting each root from x.

21. This suggests the following method of forming an equation of any degree when its roots are known:

If a, b, c, etc., are the roots of an equation, multiply together

the factors $x - a$, $x - b$, $x - c$, etc., and place the product equal to zero.

Example. Form an equation whose roots shall be 5, 7, and 8.

$$(x - 5)(x - 7)(x - 8) = 0,$$

or

$$x^3 - 20x^2 + 131x - 280 = 0,$$

which is the equation required.

It is evident that 5, 7, and 8 are roots of this equation from the fact that when $x =$ any one of them the corresponding factor is zero, and consequently the whole equation becomes zero.

Example. Form the equation whose roots shall be

$$4, -4, 2 + \sqrt{-3}, \text{ and } 2 - \sqrt{-3}.$$

The equation will be

$$(x - 4)(x + 4)[x - (2 + \sqrt{-3})][x - (2 - \sqrt{-3})] = 0.$$

The multiplication may be simplified as follows:

$$x - (2 + \sqrt{-3})$$

and $$x - (2 - \sqrt{-3})$$

may be written $$(x - 2) - \sqrt{-3}$$

and $$(x - 2) + \sqrt{-3},$$

where we have the product of the sum and difference of the quantities $(x - 2)$ and $\sqrt{-3}$.

The product will be

$$(x-2)^2-(\sqrt{-3})^2=x^2-4x+7.$$

This product multiplied by the two preceding factors becomes

$$(x-4)(x+4)(x^2-4x+7)=0,$$

or
$$x^4-4x^3-9x^2+64x-112=0.$$

Another Method.

22. *The coefficient of the second term in the equation of the nth degree is equal to the algebraic sum of the roots with their signs changed.*

The coefficient of the third term is equal to the algebraic sum of the products of all the roots, taken in sets of two.

The coefficient of the fourth term is equal to the algebraic sum of the products of all the roots, taken in sets of three, with their signs changed.

The last term is equal to the continued product of all the roots with their signs changed.

Let a, b, c, d, etc., be the values of roots of the proposed equation. The equation is formed by multiplying together the n binomial factors, $x-a$, $x-b$, $x-c$, etc., and the rule expresses the law which such multiplication must follow. Thus:

$$
\begin{array}{l}
x - a \\
x - b \\
\hline
\end{array}
$$

$$
(1) \quad \begin{array}{r|l}
x^2 - a & x + ab \\
- b &
\end{array}
$$

$$x - c$$

Multiply (1) by $x - c$.

$$
(2) \quad \begin{array}{r|r|l}
x^3 - a & x^2 + ab & x - abc \\
- b & + ac & \\
- c & + bc &
\end{array}
$$

$$x - d$$

Multiply (2) by $x - d$).

$$
(3) \quad \left.\begin{array}{r|r|r|l}
x^4 - a & x^3 + ab & x^2 - abc & x + abcd \\
- b & + ac & - abd & \\
- c & + ad & - acd & \\
- d & + bc & - bcd & \\
 & + bd & & \\
 & + cd & &
\end{array}\right\} = 0
$$

It will be seen that the coefficients in (1), (2), and (3) conform to the rule.

23. The rule of *combinations* will show how many products enter into each coefficient.

This rule may be expressed thus:

The number of combinations of n things taken q at a time is a fraction whose numerator is the continued product of q successive numbers from n downward, and whose denominator is the continued product of q successive numbers from 1 *upward.*

Example. Form an equation whose roots are

$$1, \; -2, \; 3, \text{ and } 4.$$

By the above rule the equation will be

$$\left.\begin{array}{r|l|l|l}
x^4 - 1 & x^3 + 1\times(-2) & x^2 + (-1)\times \;\; 2\times(-3) & x + (-1)\times 2 \\
+2 & \;\; +1\times 3 & \;\; +(-1)\times \;\; 2\times(-4) & \;\; \times(-3)\times(-4) \\
-3 & \;\; +1\times 4 & \;\; +(-1)\times(-3)\times(-4) & \\
-4 & \;\; -2\times 3 & \;\; + \;\; 2\times(-3)\times(-4) & \\
 & \;\; -2\times 4 & & \\
 & \;\; +3\times 4 & &
\end{array}\right\} = 0;$$

or $$x^4 - 6x^3 + 3x^2 + 26x - 24 = 0.$$

The results in Arts. 24, 25, 26 and 27 are deduced from the above rule.

24. Since the absolute term is the product of all the roots with their signs changed, it must be divisible by each root. Hence, when all the roots but one of an equation are known, that one may be found from the absolute term.

Example 1. In the equation

$$x^3 + 15x^2 + 71x + 105 = 0,$$

two roots are -5 and -3. What is the third root?

Ex. 2. In the equation

$$x^3 - x^2 - 7x + 15 = 0,$$

two roots are $2 + \sqrt{-1}$ and $2 - \sqrt{-1}$. What is the third root?

25. If the absolute term is wanting, one root of the equation is zero. For, since the absolute term is the product of all the roots with their signs changed, it cannot vanish unless one of the roots of the equation is zero.

Example. One root of the equation

$$x^3 - 4x^2 + x = 0$$

is $2 + \sqrt{3}$. What are the two remaining roots?

26. Since the coefficient of the second term is the sum of all the roots with their signs changed, it follows that when all the roots of an equation but one are known, this one may be found from the coefficient of the second term.

Example 1. If 10 and 8 are roots of the equation

$$x^3 - 17x^2 + 62x + 80 = 0,$$

show, from the second term, what the third root is.

Ex. 2. The equation

$$x^4 - 3x^3 - 54x^2 + 112x = 0$$

has four roots, two of which are 2 and -7. What are the others?

27. If the second term of an equation is wanting, the sum of the positive roots must be equal to the sum of the negative roots.

Example 1. The equation

$$x^4 - 61x^2 + 180x = 0$$

has four roots, two of which are 4 and 5. What are the others?

Ex. 2. The equation

$$x^4 - 3x^2 + 2x = 0$$

has four roots, one of which is -2. What are the other three? Verify the result.

28. In forming an equation when the roots are given, it is usually best to employ the first method when some of the roots are surds, and the second method may be employed when all of the roots are rational.

29. *If $x - a$ is an exact divisor of the first member of the general equation of the nth degree, a is a root of the equation.*

Illustration. In the equation

$$x^3 - 6x^2 + 11x - 6 = 0,$$

the first member is found by trial to be exactly divisible by $x - 1$, and this gives a quotient $x^2 - 5x + 6$. Hence

$$(x^2 - 5x + 6)(x - 1) = 0.$$

Since both members are divisible by either factor, we may divide by $x^2 - 5x + 6$, and obtain

$$x - 1 = 0.$$

Whence $x = 1$, and 1 is a root.

30. If we divide both members by $x - 1$ the result is

$$x^2 - 5x + 6 = 0,$$

an equation one degree lower than the original equation.

Thus, dividing the general equation by $x - a$, where a is a root, reduces it to the next lower degree. This is called *depressing* the equation.

31. Conversely, *If a is a root of the general equation of the nth degree, the first member is exactly divisible by $x - a$.*

Illustration. In the equation

$$x^3 - 5x^2 + 8x - 6 = 0$$

substitute 3 for x.

$$\begin{array}{rrrr|l} 1 & -5 & +8 & -6 & \underline{3} \\ & 3 & -6 & +6 & \\ \hline & -2 & +2 & 0 & \end{array}$$

Since there is no remainder, 3 is a root.

Divide $x^3 - 5x^2 + 8x - 6$ by $x - 3$.

$$\begin{array}{l|l} x^3 - 5x^2 + 8x - 6 & \underline{x - 3} \\ \underline{x^3 - 3x^2} & 1x^2 - 2x + 2 \\ \quad - 2x^2 + 8x & \\ \quad \underline{- 2x^2 + 6x} & \\ \qquad\quad 2x - 6 & \\ \qquad\quad \underline{2x - 6} & \\ \qquad\quad 0 \quad 0 & \end{array} \quad \text{or} \quad \begin{array}{l|l} 1 - 5 + 8 - 6 & \underline{1 - 3} \\ \underline{1 - 3} & 1 - 2 + 2 \\ \quad - 2 + 8 & \\ \quad \underline{- 2 + 6} & \\ \qquad + 2 - 6 & \\ \qquad \underline{+ 2 - 6} & \\ \qquad\quad 0 \quad 0 & \end{array}$$

Since there is no remainder, $x - 3$ is an exact divisor of the first member of the equation.

Abridged Method of Division.

32. It will be seen that the successive coefficients in the above quotient are the same as the successive remainders in the substitution. In general, *the short method of substitution may be used to obtain the quotient arising from the division of the general equation by $x - a$.*

Example. Divide

$$x^7 + x^6 - 14x^5 - 14x^4 + 49x^3 + 49x^2 - 36x - 36 = 0$$

by $x - 3$.

$$\begin{array}{rrrrrrrr|l} 1 & +1 & -14 & -14 & +49 & +49 & -36 & -36 & \underline{3} \\ & +3 & +12 & -6 & -60 & -33 & +48 & +36 & \\ \hline 1 & +4 & -2 & -20 & -11 & +16 & +12 & 0 & \end{array}$$

Supplying the powers of x, and remembering that the quotient will be *one degree* lower than the dividend, we may write as the result of the division the depressed equation

$$x^6 + 4x^5 - 2x^4 - 20x^3 - 11x^2 + 16x + 12 = 0.$$

Examples. Form the equations whose roots are as follows:

(1) $6,\ 8,\ 2.$

(2) $2,\ 3,\ 4,\ 5,\ 6.$

(3) $2,\ -2,\ 4 + \sqrt{7},\ 4 - \sqrt{7}.$

(4) $6 + \sqrt{-3},\ 6 - \sqrt{-3}.$

(5) $4,\ \sqrt{-2},\ -\sqrt{-2}.$

(6) $-2,\ 3,\ -4,\ 5,\ -6.$

(7) $\pm 2,\ \pm 4.$

(8) $0,\ 1,\ 2,\ 3.$

(9) $0,\ 2 \pm \sqrt{-2},\ 3 \pm \sqrt{3}.$

In each of the following equations two roots are to be found in addition to those given:

(1) $x^4 - 3x^3 - 14x^2 + 48x - 32 = 0.$

The roots given are 1 and 2.

(2) $x^3 - 4x^2 + 3x + 2 = 0.$ The root given is 2.

(3) $x^5 - 3x^4 - 16x + 48 = 0.$

The roots given are $-2, +2, +3$.

(4) $x^6 - 4x^4 - x^2 + 4 = 0.$

The roots given are $-2, +2, -1, +1$.

33. *Every equation of the nth degree containing but one unknown quantity has n roots and no more.*

This proposition is true of every equation which has any root, and depends on the assumption that every such equation is made up of factors of the form $x - a$, $x - b$, etc. An equation of this kind will contain n such factors, and may be written in the form

$$(x - a)(x - b)(x - c) \ldots\ldots (x - l) = 0.$$

This equation will be satisfied by any one of the n values

$$x = a, \quad x = b, \quad x = c, \quad \text{etc.}$$

Consequently these values are roots of the equation.

The equation has no more than n roots, because if we ascribe to x a value which is not one of the n roots $a, b, c,$ etc., this value will not cause any one of the factors of the

above equation to be zero, and the product of the several factors cannot be zero when no one of the factors is zero.

34. When we say that an equation of the nth degree has n roots we simply mean that its first member can be resolved into n factors of the form $x - a$, and each factor involves one root. Any number and indeed all of these factors may be equal. Thus the equation

$$x^3 - 6x^2 + 12x - 8 = 0$$

can be resolved into the factors $(x - 2)(x - 2)(x - 2) = 0$; whence it appears that the three roots of this equation are 2, 2, 2.

Example. Find all the roots of the following equations:

(1) $x^3 = 8$.

(2) $x^4 = 81$.

(3) $x^6 = 64$.

(4) $x^4 - 8x^2 + 16 = 0$.

35. *In an equation whose coefficients are all real, imaginary roots must occur in pairs.*

The equation is the product of n factors of the first degree. If one of the roots be of the form $\sqrt{-a^2}$ or $a\sqrt{-1}$, one of the factors will be of the form $x - a\sqrt{-1}$.

If this be multiplied by $x + a\sqrt{-1}$, the product will be $x^2 + a^2$, and no imaginary terms will be introduced into the equation. If only one factor of the equation contain an imaginary term, the expression $\sqrt{-1}$ will enter into some

coefficient; which is contrary to the hypothesis that all the coefficients are real.

Similarly, when the coefficients are all real, if a root occurs of the form $a + b\sqrt{-1}$, another root must exist of the form $a - b\sqrt{-1}$. Two factors of the equation will then be

$$\begin{array}{ll|ll} & x - (a + b\sqrt{-1}) & \text{or} & (x - a) - b\sqrt{-1} \\ \text{and} & x - (a - b\sqrt{-1}), & \text{and} & (x - a) + b\sqrt{-1}, \end{array}$$

whose product will be $(x - a)^2 + b^2$.

36. *In an equation whose coefficients are all rational, surd roots must occur in pairs.*

The coefficients being rational, a factor of the form $x - \sqrt{a}$ must occur in connection with the factor $x + \sqrt{a}$, and a pair of roots will be $\pm \sqrt{a}$.

Similarly an equation with rational coefficients may have a pair of roots of the form $a \pm \sqrt{b}$, but neither $a - \sqrt{b}$ nor $a + \sqrt{b}$ alone.

Examples.

1. One root of the equation $x^3 - 2x + 4 = 0$ is

$$1 + \sqrt{-1}.$$

What are the other roots?

2. One root of the equation $x^3 - x^2 - 7x + 15 = 0$ is

$$2 + \sqrt{-1}.$$

What are the other roots?

3. One root of the equation $x^3 + x^2 - x + 15 = 0$ is

$$1 + 2\sqrt{-1}.$$

What are the other roots?

4. One root of the equation $x^4 - 4x^3 + 4x - 1 = 0$ is

$$2 - \sqrt{3}.$$

What are the other roots?

5. Two roots of the equation

$$x^8 + 2x^6 + 4x^5 + 4x^4 - 8x^2 - 16x - 32 = 0$$

are $\qquad -1 + \sqrt{-1} \quad$ and $\quad 1 - \sqrt{-3}$.

What are the other six roots?

Principles to be Observed in Plotting.

37. Since an equation of the nth degree can have no more than n roots, its curve cannot cross the *x-axis* more than n times.

Hence a curve of the nth degree cannot have more than $n - 1$ *elbows.*

We use the term *elbow* to designate a point of the curve where the curve, from going either towards or from the *x-axis*, turns to go in the opposite direction, as at A, B, C, D, in Fig. 8.

Any line parallel to the *y-axis* can cross the curve but once.

No straight line can cut the curve of the nth degree more than n times.

A curve of an odd degree has its extremities on opposite sides of the *x-axis*, and a curve of an even degree has its extremities on the same side of the *x-axis*.

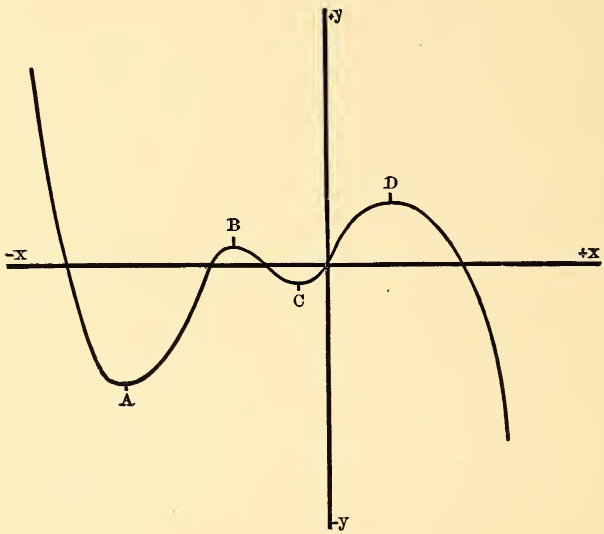

FIG. 8.

If an equation contains no absolute term, its curve passes through the origin; that is, one root of the equation is equal to zero.

Equal and Imaginary Roots.

38. If we plot the equation

$$y = x^2 - 4x - 5,$$

its curve will be found to intersect the *x-axis* at the points $x = -1$ and $x = 5$. (Fig. 9.)

If the absolute term of this equation be increased by 8, the equation becomes

$$y = x^2 - 4x + 3,$$

and each value of y in the table is therefore increased by 8. Each point of the curve is therefore raised up 8 units, while

x	y
0	-5
$+1$	-8
$+2$	-9
$+3$	-8
$+4$	-5
$+5$	0
$+6$	$+7$
-1	0
-2	$+7$

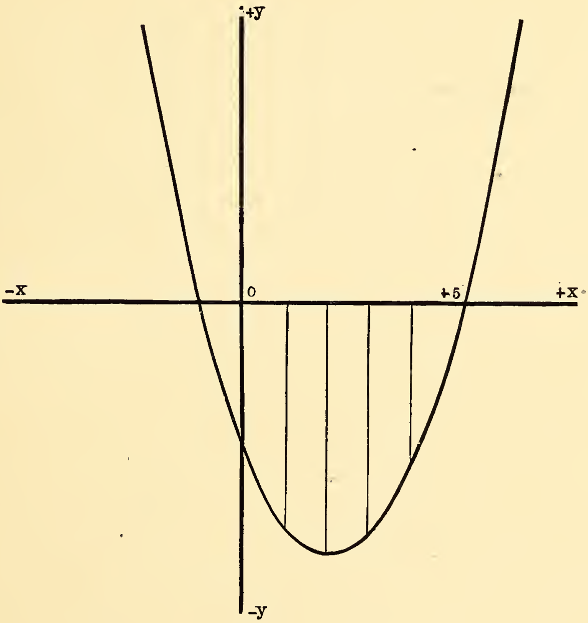

Fig. 9.

the shape of the curve is unchanged. Its intersections with the *x-axis* are brought nearer together, and the roots of

$$x^2 - 4x + 3 = 0$$

are $+1$ and $+3$. (Fig. 10.)

If the absolute term be still more increased, the shape of the curve will remain unchanged, but the curve will be raised

x	y
0	$+3$
$+1$	0
$+2$	-1
$+3$	0
$+4$	$+3$
$+5$	$+8$
-1	$+8$
-2	$+15$

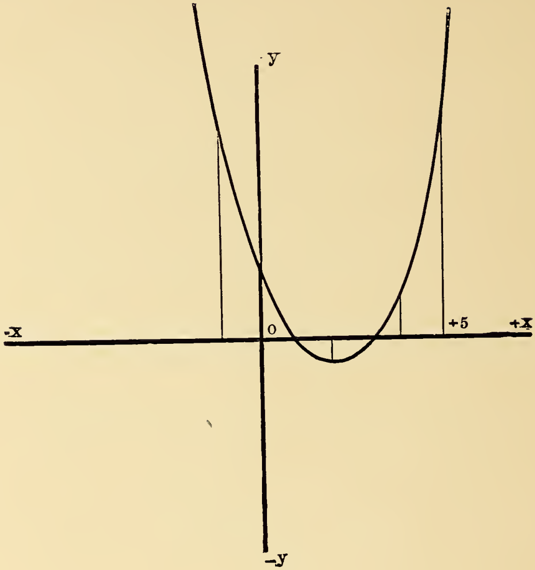

FIG. 10.

x	y
0	$+4$
$+1$	$+1$
$+2$	0
$+3$	$+1$
$+4$	$+4$
$+5$	$+9$
-1	$+9$

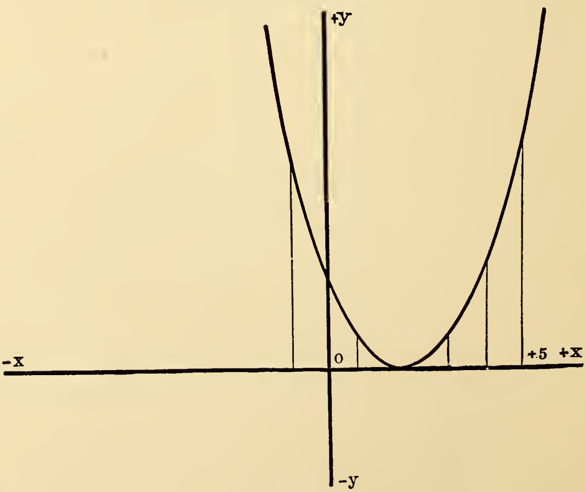

FIG. 11.

higher and the intersections with the *x-axis* will come nearer and nearer together, until, when the equation becomes

$$y = x^2 - 4x + 4,$$

the two intersections will coincide and the elbow of the curve will rest on the *x-axis*. The roots of this equation are found to be $+2$ and $+2$. (Fig. 11.)

If we increase the absolute term beyond this point, the curve will lie wholly above the *x-axis*, and the roots will be imaginary. Thus the roots of the equation

$$x^2 - 4x + 7 = 0$$

are
$$2 \pm \sqrt{-3},$$

and the plot of
$$y = x^2 - 4x + 7$$

will be—

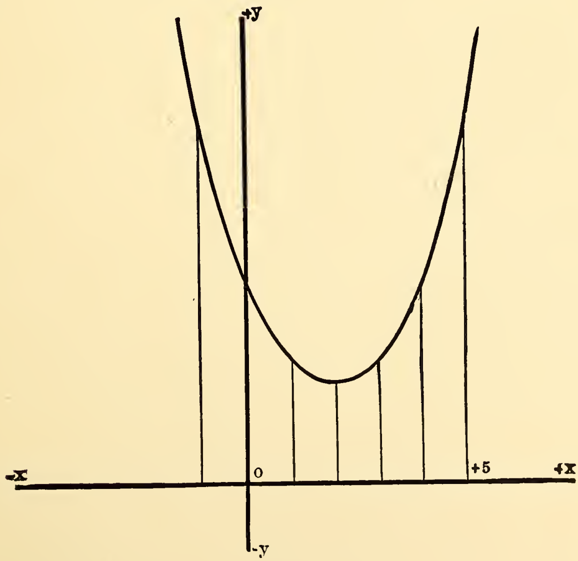

FIG. 12.

It will be found in plotting equations of any degree that the relation of each elbow of the curve to the *x-axis* will change in the same way as above by changing the absolute term of the equation; that is, changing the absolute term raises or lowers every point of the curve by the same amount without changing the form of the curve.

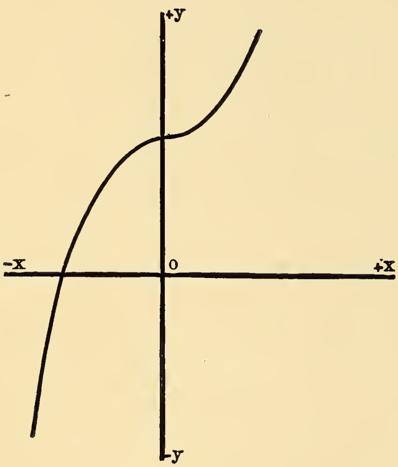

FIG. 13.

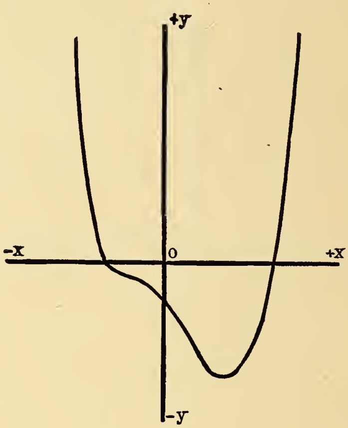

FIG. 14.

39. When an elbow of the curve rests on the *x-axis*, the equation has a pair of equal roots.

40. From the above illustration it may be seen why imaginary roots occur in pairs. For as the curve is raised or lowered whenever intersections disappear, two must disappear together.

41. In general whenever a curve of the *n*th degree inter-

sects the *x-axis* less than n times, some of its roots must be equal or imaginary.

The student must not expect always to find $n - 1$ distinct elbows in the curve of the nth degree, for reasons to be explained in the next chapter. Thus Figs. 13 and 14, page 40, are examples of curves of the third and fourth degree which have each two less than $n - 1$ elbows.

Plot the following equations:

(1) $y = x^3 + 3x^2 + 4x - 1.$

(2) $y = x^5 - 2x^4 - x + 2.$

(3) $y = x^4 - 4.$

42. The directions given for plotting will enable one to find, by measuring the distances from the origin along the *x axis* to its intersections with the curve, the exact or the approximate values of the real roots of an equation.

Example. Plot the equation

$$y = x^4 - 3x^3 - 15x^2 + 19x + 57,$$

and find, by measurement, the integral part of each of the roots of

$$x^4 - 3x^3 - 15x^2 + 19x + 57 = 0.$$

The following is the plot:

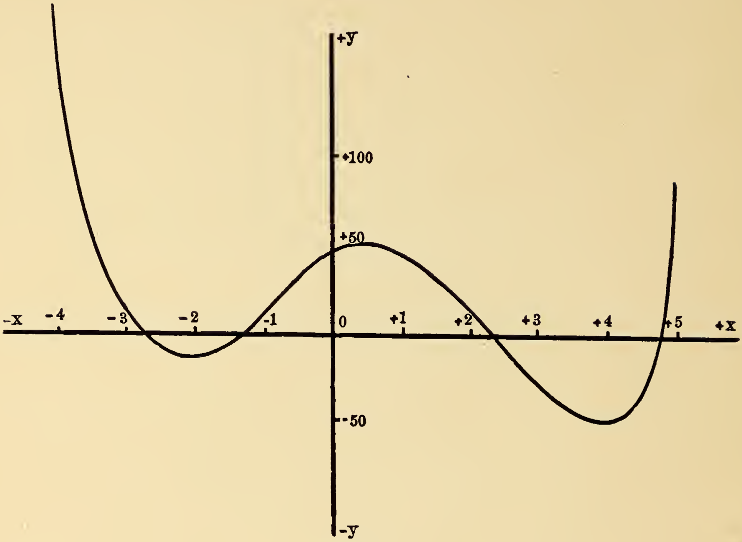

FIG. 15.

Examples. Make plots for the following equations, and find by measurement the values of their real roots:

(1) $x^2 - 6x + 10 = 0.$

(2) $x^3 + 12x^2 + 50x + 48 = 0.$

(3) $x^3 + 12x^2 + 44x + 48 = 0.$

(4) $x^4 + 3x^3 - 12x^2 - 20x + 53 = 0.$

(5) $x^4 + 8x^3 + 24x^2 + 32x + 16 = 0.$

(6) $x^5 - 10\,x^4 + 41x^3 - 86x^2 + 52x - 40 = 0.$

43. The roots of equations of a higher degree than the second, except in some special cases, are usually determined, in practice, approximately by trial. When no plot is used, it is necessary to assume a series of values of x, and by sub-

stitution find how nearly each one will satisfy the equation; that is, reduce the first member to zero. The results of these substitutions will enable us to find approximate values of the real roots. Now the plot of an equation is a graphical representation of the results of a systematic series of substitutions, and enables us to determine at a glance the approximate situation of all the real roots. It enables us also to detect any errors that may occur in the substitutions; for, if the substitutions are correct, the line connecting the points so determined will always be a smooth curve. The plot also indicates the presence of imaginary roots, and, above all, enables us to interpret geometrically the various propositions in the theory of equations. In short, the plot bears the same relation to the study of equations that a map bears to the study of geography.

To solve an equation we seek only for such values of x as will reduce the first member to zero; that is, will make y equal to zero; but the plot is a representation of every possible value of y, and therefore, when we know the nature of the curve in the neighborhood of its intersections from the plot, we have a guide for determining the approximation to the roots of the equation.

The simplest method in theory for obtaining the value of a root to any degree of approximation is the method of Double Position.

Approximation by Double Position.

44. RULE. *Assume two different values of x, substitute each in the given equation, and note the* ERRORS *of the results. Then as the difference of the errors is to the difference of the assumed numbers, so is either error to the correction of its assumed number.*

This rule assumes that the errors of the result are proportional to the errors of the assumed numbers. This supposition is not entirely correct except in equations of the first degree; but if we employ numbers near to the true values, the error of this supposition is generally not very great, and the error becomes less and less the farther we carry the approximation.

Illustration. Find one root of the equation

$$x^3 + x^2 + x - 115 = 0.$$

Plotting the equation

$$y = x^3 + x^2 + x - 115,$$

it is seen that the curve crosses the *x axis* between $x = +4$ and $x = +5$. Therefore 4 and 5 may be taken as the two *assumed numbers*, and the corresponding values of y, viz. -31 and $+40$, will be their *errors.*

Connect the points A and E, and draw the line ED parallel

to the *x-axis*, meeting AF produced in D. Since the triangles ADE and BEC are similar, we have the proportion

$$AD : ED :: BE : BC;$$

or, $$71 : 1 :: 31 : (.4+).$$

Hence $$OC = 4.4+.$$

Here AD = difference of *errors;*

ED = difference of assumed numbers;

BE = first error;

OB = first assumed number;

and BC = correction to be applied to OB.

x	y
0	− 115
+ 1	− 112
+ 2	− 101
+ 3	− 76
+ 4	− 31
+ 5	+ 40
+ 6	+ 143
− 1	− 116
− 2	− 121
− 3	− 136
− 4	− 168

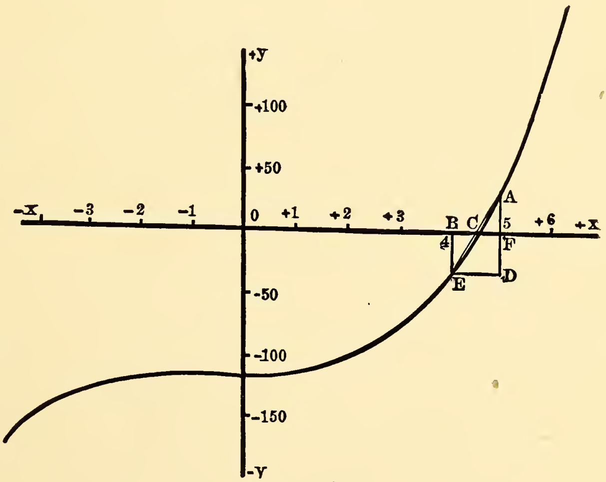

FIG. 16.

The point C is not the intersection of the *curve* with the *x-axis*, but, since in the short distance AE the curve differs but little from a straight line, the point C is very near the

required intersection, and OC is an approximate value of the root.

45. Now if the equation had been of the first degree, the line joining any two points, as A and E, would be a straight line, since in such an equation the values of x and y are proportional, and hence the point C would be at the intersection required, and the solution would be exact.

46. For an equation of a degree higher than the first, the portion of the curve between any two assumed points will not be a straight line, and hence the result obtained by Double Position will only be approximately correct.

47. The approximation will usually be closer if the two assumed points lie on opposite sides of the crossing.

Second Approximation.

48. Substituting 4.4 for x in the equation gives $y = -6.06$. This shows that the root is greater than 4.4.

Substituting 4.5 for x gives $y = +0.88$.

Therefore the point where the curve crosses the *x-axis* lies between

$$x = 4.4 \qquad \text{and} \qquad x = 4.5.$$

The following is a plot of this portion of the curve on an enlarged scale:

x	y
$+4.4$	-6.06
$+4.5$	$+0.88$

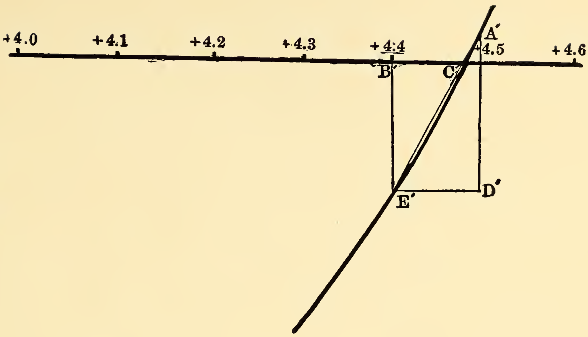

FIG. 17.

Drawing the lines to form the two triangles as in the first approximation, we have by similar triangles the proportion

$$A'D' : E'D' :: B'E' : B'C';$$

$$6.94 : .1 :: 6.06 : (.087).$$

Adding this value to the approximate value of the root already found gives

$$x = 4.487+$$

In this way the approximation may be carried on as far as desired.

49. In the second approximation it will be desirable to find two values of x on opposite sides of the crossing, which differ from each other only by one tenth.

Examples. Find, by the plot and Double Position, approximate values of the real roots of the following equations:

(1) $x^3 + 2x^2 + 3x - 50 = 0.$

(2) $x^3 + 2x^2 - 23x - 70 = 0.$

(3) $x^5 - 81x + 40 = 0.$

(4) $x^4 - 55x^2 - 30x + 400 = 0.$

CHAPTER III.

DIRECTION OF THE CURVE. THE DERIVATIVE. EQUAL ROOTS.

50. We may illustrate the idea of the direction of a curve by supposing the course of a stream to be such, that, when a boat is carried down it by the current, the centre of gravity of the boat will describe the curve of an algebraic equation. The course of the boat at any point is the direction of a line drawn through the boat from stem to stern. This line is always a tangent to the curve on which the centre of gravity is moving, and gives the *direction* of the curve at any point. Now assume the x and y axes of the curve to be drawn, and consider the *x-axis* as an East and West line and the *y-axis* a North and South line.

If the direction of the tangent line is Northeasterly and Southwesterly, we shall call its inclination to the *x-axis* positive, and if Northwesterly and Southeasterly, negative.

51. We measure the inclination of the tangent as follows: Drop a perpendicular on the *x-axis* from the point A and draw the tangent AC. Then the ratio $\frac{AB}{BC}$ measures this in-

clination. Suppose (Fig. 18) AB is twice BC, then the measure of the inclination is 2. Moving down the curve towards the

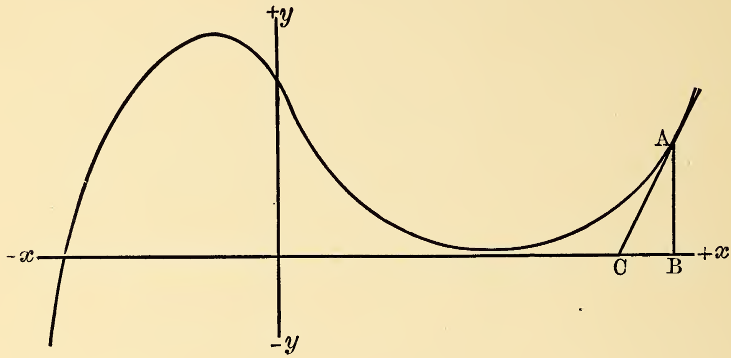

FIG. 18.

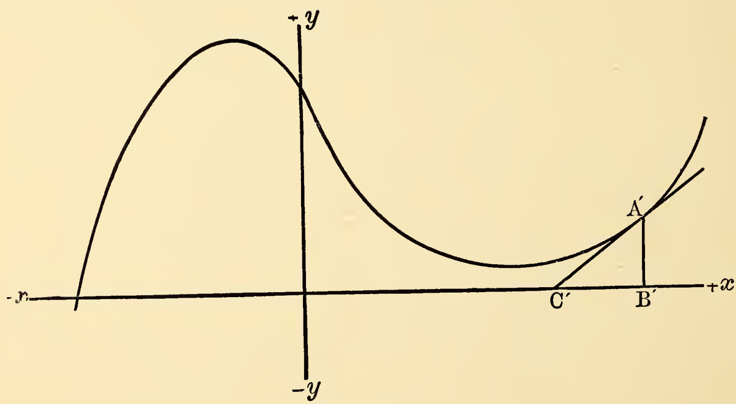

FIG. 19.

elbow, the inclination grows less and less. At A' (Fig. 19), $\frac{A'B'}{C'B'} = 1$; at the elbow (Fig. 20), or A'', the tangent is parallel to the *x-axis* and the measure of the inclination is

$\frac{A''B''}{\infty} = 0$; continuing past the elbow, the inclination becomes negative (Fig. 21) and remains so until we reach the second elbow, when it again becomes zero and then becomes

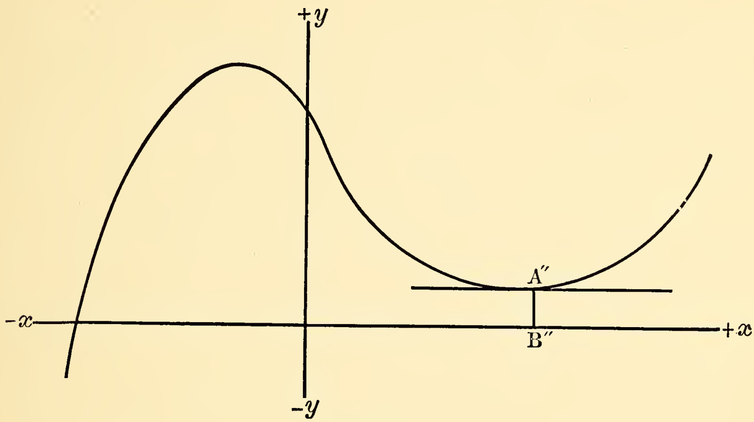

Fig. 20.

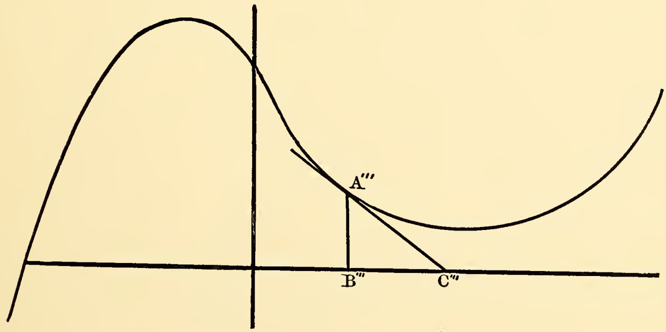

Fig. 21.

positive. The algebraic expression which gives the measure of the inclination of the tangent, or the direction of the curve, is called a *derivative*.

52. The derivative of an equation may be found by multiplying each term of the equation by the exponent of x in that term and diminishing its exponent by unity.

Example. What is the derivative of the equation

$$x^3 + 3x^2 + 5x - 178 = 0?$$

Ans. $3x^2 + 6x + 5$.

What is the inclination of the tangent at the point of the curve for which $x = 2$? Ans. 29.

53. The following example will show how the measure of the inclination of the tangent is given by the derivative of the equation:

$$y = x^3 + 3x^2 + 5x - 178.$$

On the curve of the above equation take two points B and B' near each other, and draw a secant $BB'C$. Call the values of the x and the y of the point B, x' and y' respectively, and let dx and dy be the differences of the x's and y's of the two points. Then the x and y of B' are $x' + dx$ and $y' - dy$. Since the equation expresses the relation between the value of the x and the value of the y for any point, we have by substitution of the two sets of values—

For B', $\quad y' - dy = x'^3 + 3x'^2 + 5x' - 178 + (3x'^2 + 6x' + 5)dx + (3x' + 3)(dx)^2 + (dx)^3$

For B, $\quad y' = x'^3 + 3x'^2 + 5x' - 178$

Subtracting, $\quad -dy = (3x'^2 + 6x' + 5)dx + (3x' + 3)(dx)^2 + (dx)^3$

Dividing by dx, $\quad -\dfrac{dy}{dx} = 3x'^2 + 6x' + 5 + (3x' + 3)dx + (dx)^2.$

Now by similar triangles,

$$\frac{dy}{dx} = \frac{AB}{AC};$$

whence

$$-\frac{AB}{AC} = 3x'^2 + 6x' + 5 + (3x' + 3)dx + (dx)^2.$$

But this relation holds true no matter how small the distance between the points; and as the points are brought nearer and nearer together, the value of $\frac{AB}{AC}$ changes only as

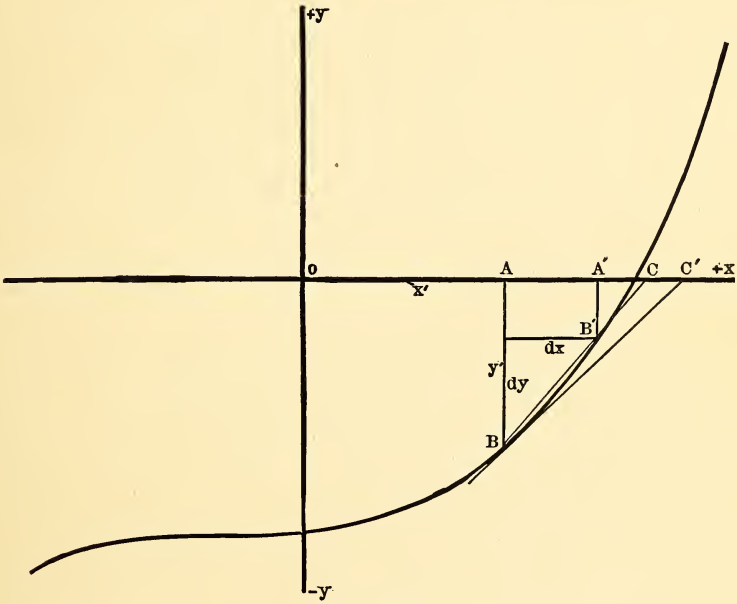

FIG. 22.

the value of the terms containing dx and $(dx)^2$ change. Now if B' and B are made to coincide, $\frac{AB}{AC}$ becomes $\frac{AB}{AC'}$, and dx and $(dx)^2$ become zero; hence, when BC' is a tangent line, $-\frac{AB}{AC'} = 3x'^2 + 6x' + 5$. But $-\frac{AB}{AC'}$ measures the inclina-

tion of this tangent line, and $3x'^2 + 6x' + 5$ is the derivative of the equation, and therefore in this case the inclination of the tangent is measured by the derivative.

In the same way it may be shown for any like equation.

54. *The Derivative Curve.* If we put the derivative of an equation equal to y' and then plot the resulting equation on

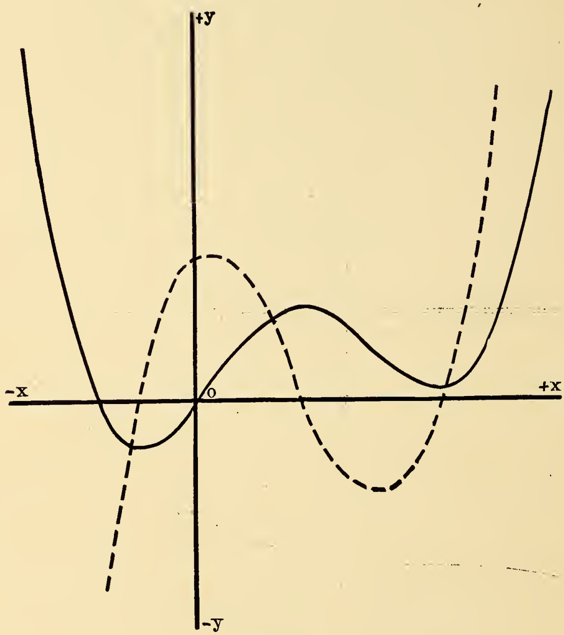

FIG. 23.

the same scale and the same axes we employed in plotting the original equation, any value of x in the derivative will give a value of y' which is the measure of the inclination of the tangent line drawn to the original curve at the point corresponding to this value of x. When the inclination is posi-

tive, the derivative curve will be above the x-*axis*; when negative, below; and when the direction is parallel to the x-*axis*—that is, at each elbow—the derivative curve will cross the x-*axis*.

55. From this construction we have the following rule for finding the position of the elbows of a curve:

Place the derivative equal to zero and solve the equation.

There will be elbows in the curve of the original equation for those values of x which are real roots of the derivative equation. For, the values of x for each elbow of the original curve give the points where the derivative curve crosses the x-*axis;* but as the derivative curve shows only the *direction* of the original curve without regard to its distance above or below the x-*axis*, it follows that the elbow of the curve may be above, below, or at the point where the derivative crosses the x-*axis*.

56. *A Pair of Equal Roots.* We have seen in Art. 38 that by changing the value of the absolute term in an equation we may raise or lower the curve so as to bring any elbow to rest on the x-*axis* and thus obtain a *pair of equal roots*.

Since, when an elbow of the curve touches the x-*axis*, the derivative curve passes through this point of tangency, then, if the distance of this point from the origin be a, each equation will have a root equal to a, and therefore will be divisible by $x - a$. *Hence, when an equation has a pair of equal roots, there will be a common divisor between the equation and its first derivative of the form* $x - a$.

Example. Find the equal roots in the equation

$$x^3 + 8x^2 + 16x = 0.$$

The derivative of this equation is

$$3x^2 + 16x + 16.$$

The greatest common divisor between the equation and the derivative is $x + 4$. Putting this equal to 0, we have $x = -4$ for the value of each of the equal roots. (Fig. 24.)

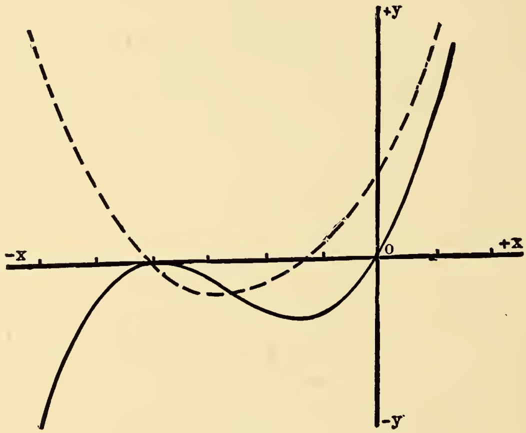

FIG. 24.

Example. Find the equal roots in the equation

$$x^4 - 2x^2 + 1 = 0.$$

The greatest common divisor between the equation and

NOTE.—In this chapter the dotted curves are the plots of the derivative equations.

its first derivative is $x^2 - 1$, whence $x = +1$ and $x = -1$ will each give a pair of equal roots. (Fig. 25.)

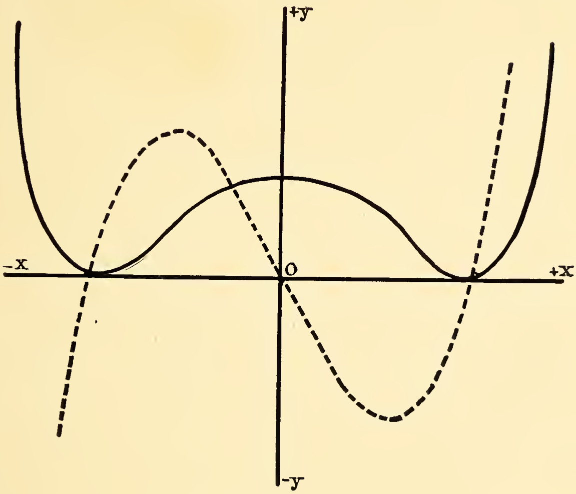

FIG. 25.

57. *Three Equal Roots.* In order that an equation may have three equal roots, it is necessary that three of its intersections should be made to coincide. Thus the equation $x^3 - x = 0$ has for its curve Fig. 26, whose intersections with the *x-axis* are at 1, 0, -1. Using the illustration of the boat, we know of course that its direction at each elbow will be parallel to the *x-axis*. If we make the coefficient of x smaller and smaller, the two elbows will approach nearer and nearer to each other. For example, in $x^3 - \frac{1}{4}x = 0$ the roots are $\frac{1}{2}$, 0, $-\frac{1}{2}$. (Fig. 27.)

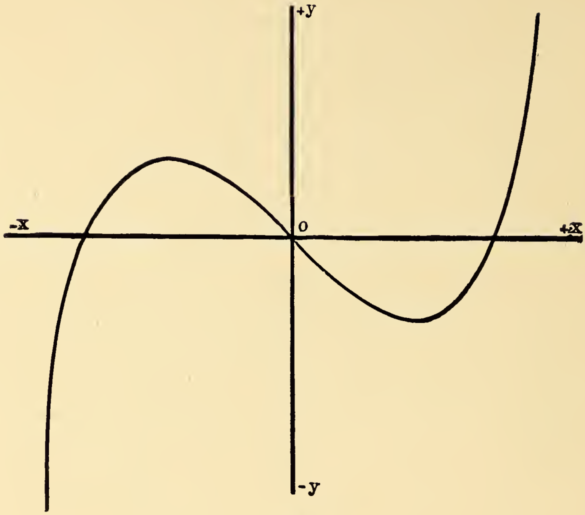

FIG. 26.

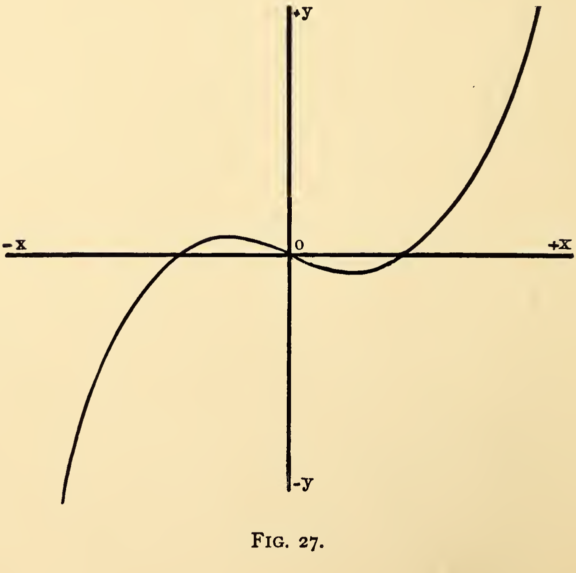

FIG. 27.

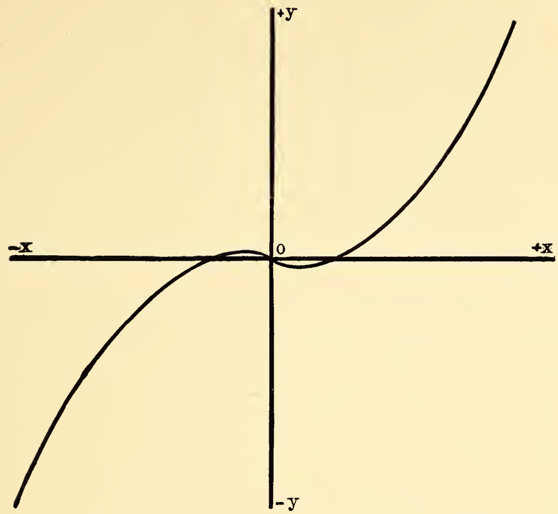

FIG. 28.

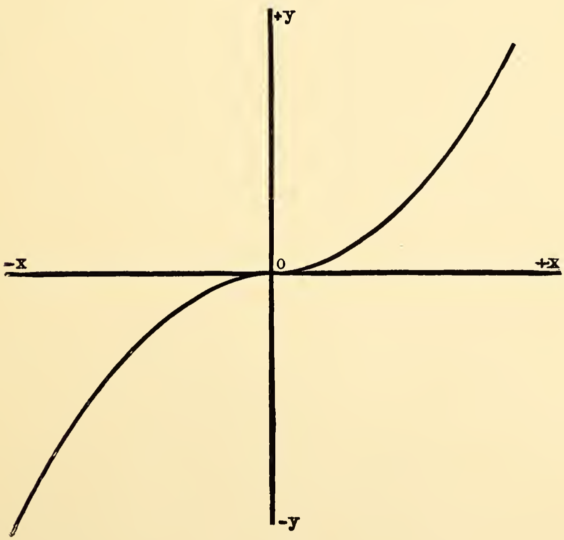

FIG 29.

In $x^3 - \frac{1}{100}x = 0$ the roots are $\frac{1}{10}$, 0, $-\frac{1}{10}$ (Fig. 28). Now when the coefficient of x becomes 0 (Fig. 29), these two elbows fade out altogether at the point 0.

When the curve crosses the *x-axis*, the direction of the boat will be along the *x-axis* at that point, and the three roots which had been approaching the same value will coincide. While the elbows of the curve come closer and closer together the intersections of the derivative curve approach each other nearer and nearer, and at the point of coincidence we have an elbow of the derivative curve resting on the x axis, and consequently the derivative equation has at this point a pair of equal roots. Thus it appears that if a is the common root of an equation and its derivative at such a point as this, the original equation contains three factors each $x - a$, and the derivative two factors $x - a$. Hence the greatest common divisor of the equation and its derivative will be a perfect square; and this is the condition of three equal roots.

Example. Find the equal roots of the equation

$$x^4 - 6x^2 - 8x - 3 = 0. \qquad \text{(Fig. 30.)}$$

58. In a way similar to that employed in explaining three equal roots, it may be shown that in the curve of the equation $x^4 - ax^2 = 0$ the elbows approach each other as the coefficient of x^2 is diminished, and when the equation becomes $x^4 = 0$ we shall have four equal roots. (Fig. 31.)

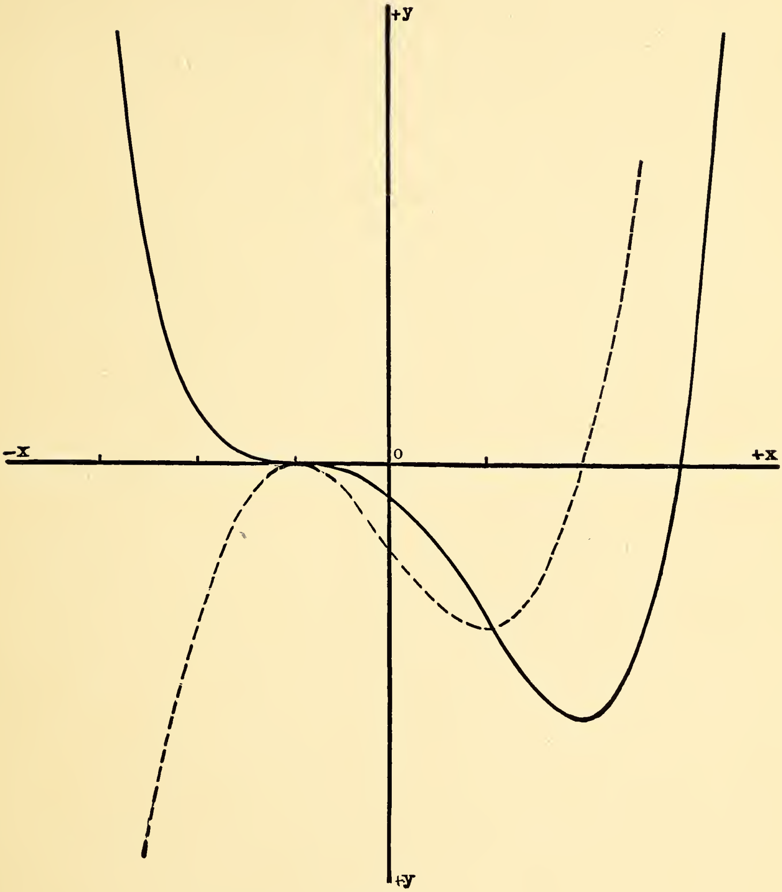

FIG. 30.

The plot here given shows the curves of the two equations on the same scale and the same lines of reference.

When an equation has four equal roots, its derivative will have three equal roots, and hence the greatest common divisor in such case will be a perfect cube.

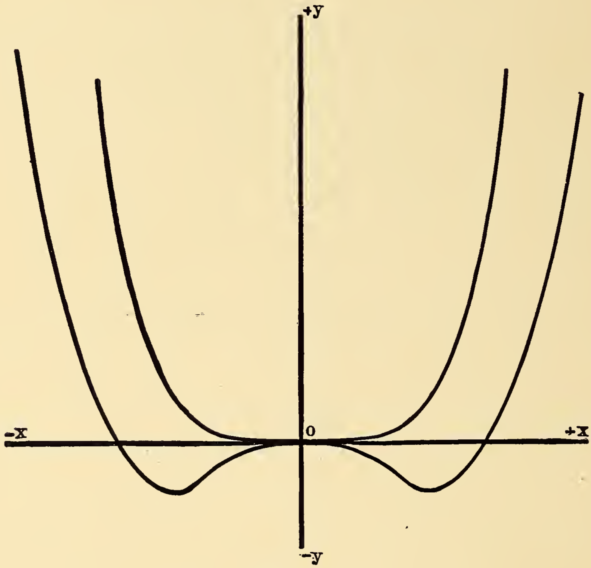

FIG. 31.

59. This principle may be extended to equations having any number of equal roots. Hence, for finding the equal roots in any equation, we have the following

RULE. *Find the greatest common divisor between the equation aud its derivative. If there is no common divisor, the equation has no equal roots. If there is a common divisor, place this equal to zero, and solve the resulting equation.*

60. If the greatest common divisor contains a factor of the form $x - a$ raised to the nth power, the given equation will contain $n + 1$ equal roots.

61. It is not usually possible to determine from its plot alone whether an equation has equal roots or not, although there is a tendency of the curve to rest for a longer distance on the x axis the greater the number of its equal roots. But however great this tendency, the curve can only coincide with the axis at one point, no matter how many equal roots there may be. That is, the boat (see Art. 50), in the case of equal roots, can coincide in position and direction with the *x-axis* at only one point. Generally the number of equal roots can only be determined by the rule.

Plot the following examples, using the same scale for x and y and assuming in each for values of x:

$$0, \quad \pm \tfrac{1}{4}, \quad \pm \tfrac{1}{2}, \quad \pm \tfrac{3}{4}, \quad \pm 1, \quad \pm 1\tfrac{1}{4}, \quad \pm 1\tfrac{1}{2}.$$

$$y = x, \qquad y = x + 1.$$

$$y = x^2, \qquad y = (x - 1)^2.$$

$$y = x^3, \qquad y = (x - 1)^3.$$

$$y = x^4, \qquad y = (x - 1)^4.$$

$$y = x^5, \qquad y = (x - 1)^5.$$

$$y = x^6, \qquad y = (x - 1)^6.$$

Interpretation of Imaginary Roots of Derivative Equations.

62. We have seen in the equation $x^3 - x = 0$ that when the coefficient of x is continually diminished, the roots approach each other nearer and nearer until, when the coefficient of x becomes zero, we have the case of three equal roots. If now we give to x a positive coefficient, no matter how small, the roots of the derivative equation will become imaginary, and this shows that the direction of the curve of the original equation is nowhere parallel to the *x-axis*, and therefore this curve has no elbows.

In general, there will be as many elbows in a given curve as there are *real* roots in the derivative equation.

Examples.

1. Find the equal roots of the equation

$$x^3 - 13x^2 + 55x - 75 = 0.$$

2. Find the equal roots of the equation

$$x^3 - 7x^2 + 16x - 12 = 0.$$

3. Find the equal roots of the equation

$$x^5 - 3x^4 - 8x^3 + 24x^2 + 16x - 48 = 0.$$

4. Find the equal roots of the equation

$$x^3 - 6x^2 + 12x - 8 = 0.$$

5. Find how many elbows, and where they are situated, in the curve of the equation

$$x^3 - 3x^2 - 9x + 37 = 0.$$

6. Find the equal roots in the equation

$$x^5 - 4x^4 - 2x^3 - 4x^2 + x = 0.$$

7. Find how many elbows, and where they are situated, in the curve of the equation

$$x^4 + 16x^3 + 90x^2 + 200x + 125 = 0.$$

CHAPTER IV.

HORNER'S METHOD OF APPROXIMATION.

63. The method of approximating to the numerical values of the roots of an equation by Double Position becomes very laborious if the approximation is carried beyond one or two decimal places. The method explained in this chapter enables us to carry on the approximation to any number of decimal places easily and systematically, and furnishes at each stage a test of the accuracy of the work. It consists in calculating the coefficients of a series of equations in each of which the roots are less than the roots of the preceding one. The roots of the *second* of these equations are less than those of the *first* by the number of units in the required root. The roots of the *third* are less than those of the *second* by the value of the first decimal place of the required root. The roots of the *fourth* are less than those of the *third* by the value of the second decimal place of the required root, and so on. Each one of these transformations is interpreted on the plot by moving the origin along the *x-axis* an amount equal to the corresponding diminution of the value of the root.

64. *If in the plot of an equation the origin be moved to the right, along the x-axis, a distance equal to a, the equation itself may be changed to correspond to the new position of the origin, by substituting in it $x' + a$ for x;* and, conversely, *if $x' + a$ be substituted for x in an equation, the form of the curve*

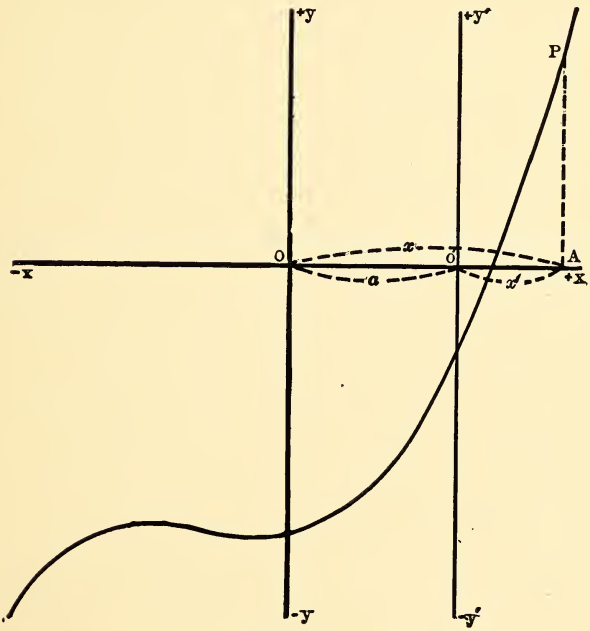

FIG. 32.

will be unchanged, but the origin will be moved along the x-axis a units to the right.

For example, in the curve of the equation

$$x^3 + 3x^2 + 2x - 31 = 0$$

the value of x for any point as P is the distance OA. If now the origin be removed along the *x-axis a* units to the right

or to the position O', it is evident that for the same point P of the curve, the value of x reckoned from O' will be a units less than if it were reckoned from the point O; but the value of y will remain the same. Calling $O'A = x'$, we have

$$OA = O'A + OO',$$

or

$$x = x' + a,$$

and therefore $x' + a$ must be substituted for x in the given equation in order to obtain the equation of the curve referred to the new origin O'.

In the equation

$$y = x^3 + 3x^2 + 2x - 31$$

the substitution of $x' + a$ for x may be made as follows:

$$\begin{array}{rl|l|l}
x^3 = & x'^3 + 3a & x'^2 + 3a^2 & x' + a^3 \\
+3x^2 = & 3 & +6a & +3a^2 \\
+2x = & & 2 & +2a \\
-31 = & & & -31 \\
\hline
\end{array}$$

$$y = x^3 + 3x^2 + 2x - 31 = x'^3 + (3a+3)x'^2 + (3a^2+6a+2)x' + a^3 + 3a^2 + 2a - 31$$

The same result may be obtained as follows:

$$\begin{array}{llll|l}
1 & +3 & +2 & -31 & a \\
 & a & a^2 + 3a & a^3 + 3a^2 + 2a & \\
\hline
1 & a+3 & a^2 + 3a + 2 & a^3 + 3a^2 + 2a - 31 & \\
 & a & 2a^2 + 3a & & \\
\hline
1 & 2a+3 & 3a^2 + 6a + 2 & & \\
 & a & & & \\
\hline
1 & 3a+3 & & &
\end{array}$$

By this last method we write the coefficients of the equation and substitute a as in the ordinary method of substitution, omitting the remainder in each succeeding operation. The several remainders are the coefficients of the successive powers of x' as found in the above substitution.

65. The first remainder, $a^3 + 3a^2 + 2a - 31$, is the absolute term of the new equation, and is therefore the distance $O'B$. (Fig. 33.)

66. The second remainder, it will be seen, gives the value of the derivative of the equation, and is the ratio $\frac{O'B}{O'C}$ (Fig. 33), and gives the direction of the curve at the point B. (See Art. 53.)

67. *Application of the preceding principle to the problem of finding the approximate numerical value of a real root of the equation*

$$x^3 + 3x^2 + 2x - 31 = 0.$$

The plot of this equation shows that one root lies between $+2$ and $+3$. Remove the origin to a point two units to the right on the *x-axis*, or to O'. Now, by the preceding article, if we put $x' + 2$ for x, the corresponding equation in terms of x' will be $x'^3 + 9x'^2 + 26x' - 7 = 0$. And this equation plotted from O' as an origin will give the same curve as the original equation plotted from O as an origin.

1	$+3$	$+2$	-31	$\underline{\vert\, 2}$
	2	10	24	
1	$+5$	$+12$	$-7 = O'B$	
	2	14		
1	$+7$	$+26 = -\dfrac{O'B}{O'C}$		
	2			
1	$+9$			

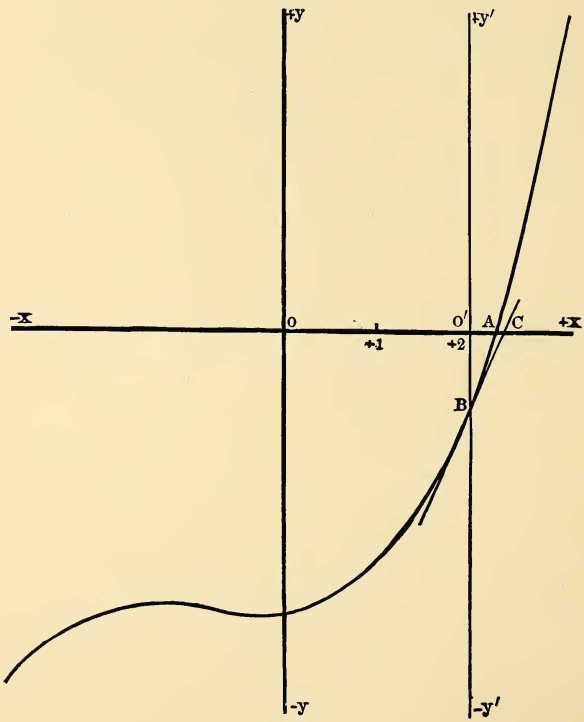

Fig. 33.

The distance OO' ($= 2$) is the value of the integral part of the root of the original equation, and, since removing the

origin does not change the form of the curve, it follows that the distance $O'A$ will be the value of the root of the new equation, which root is less than unity.

68. The value of this root might be found to the nearest tenth by forming a table of values in which x is assumed to be .1, .2, .3, .4, etc. etc. The highest number of tenths which would give a negative value for y would be the required figure of the root.

69. Sometimes it is necessary to resort to this method in obtaining the tenth's place, especially when an elbow of the curve is very close to the crossing or when several roots are very nearly equal to each other.

70. We may usually, however, obtain the value of the decimal place as follows: In Fig. 33, BC is a tangent to the curve at the point B. Then, by Art. 67,

$$O'B = -7,$$

and

$$-\frac{O'B}{O'C} = 26.$$

$O'C$ is found by dividing $O'B$ by $-\frac{O'B}{O'C}$; that is,

$$O'C = \frac{7}{26} = .2 \text{ plus a remainder.}$$

Now the tangent BC is so nearly coincident with the curve in this neighborhood that the distance $O'C$ cannot differ much from $O'A$, and hence the quotient obtained is very nearly the true value of $O'A$.

Substituting .2 in the new equation we get -1.432 for a remainder; .3, however, gives a positive remainder. Hence $O'A$ lies between .2 and .3.

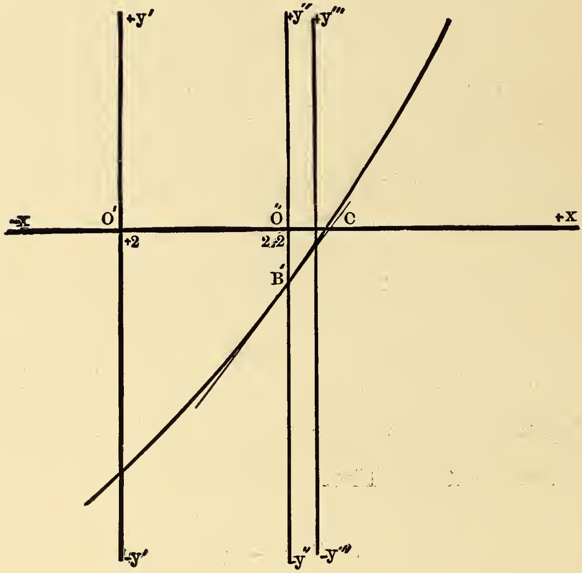

FIG. 34.

We may now remove the origin to a point O'', .2 to the right of O', and transform the equation

$$x'^3 + 9x'^2 + 26x' - 7 = 0$$

to correspond to this new change of origin by substituting

$$x'' + .2 \text{ for } x'.$$

Thus:

$$
\begin{array}{llll}
1+9 & +26 & -7 & \underline{|\ .2} \\
\quad .2 & \quad 1.84 & \quad 5.568 & \\
\hline
1+9.2 & +27.84 & -1.432 = O''B' & \\
\quad .2 & \quad 1.88 & & \\
\hline
1+9.4 & +29.72 = -\dfrac{O''B'}{O''C'} & & \\
\quad .2 & & & \\
\hline
1+9.6 & & &
\end{array}
$$

Or $$x^{3''} + 9.6x^{2''} + 29.72x'' - 1.432 = 0.$$

Proceeding as before, we obtain for the third figure of the root .04. We may carry the approximation in this way to any desired extent.

71. Now if we add together the several distances which we have removed the origin to the right from the point O, we shall obtain an approximation to the value of $O'A$, or to the root of the original equation.

The form of the calculation is given in full for two decimal places.

$$
\begin{array}{llll}
1+3 & +2 & -31 & \underline{|\ 2.24} \\
\quad 2 & \quad 10 & \quad 24 & \\
\hline
1+5 & +12 & -7 = O'B = \text{1st dividend.} & \\
\quad 2 & \quad 14 & & \\
\hline
1+7 & +26 & = -\dfrac{O'B}{O'C} = \text{1st divisor.} & \\
\quad 2 & & & \\
\hline
1+9 & +26 & -7 &
\end{array}
$$

(Carried forward.)

$1 + 9$	$+ 26$	$- 7$	
$.2$	1.84	5.568	
$1 + 9.2$	$+ 27.84$	$- 1.432$	$= O''B' =$ 2d dividend.
$.2$	1.88		
$1 + 9.4$	$+ 29.72$		$= -\dfrac{O''B'}{O''C'} =$ 2d divisor.
$.2$			
$1 + 9.6$	$+ 29.72$	$- 1.432$	
$.04$	$.3856$	1.204224	
$1 + 9.64$	$+ 30.1056$	$- .227776$	$= O'''B'' =$ 3d dividend.
$.04$	$.3872$		
$1 + 9.68$	$+ 30.4928$		$= -\dfrac{O'''B''}{O'''C''} =$ 3d divisor.
$.04$			
$1 + 9.72$			

Examples.

1. Find the real root of the equation

$$x^3 + 3x^2 + 5x = 178.$$

Ans. $4.5388 +$.

2. Find one root of the equation

$$x^5 + 2x^4 + 3x^3 + 4x^2 + 5x = 20.$$

Ans. $1.1257 +$.

3. Find the real root of the equation

$$2x^3 + 3x^2 = 850.$$

Ans. 7.0502562 +.

CASES WHERE THE RESULT OF THE DIVISION AFFORDS NO RELIABLE INFORMATION IN REGARD TO THE CORRESPONDING FIGURE OF THE ROOT.

72. FIRST CASE. *Where the divisor is zero.*

Take the equation

$$x^3 - 9x^2 + 24x - 17 = 0.$$

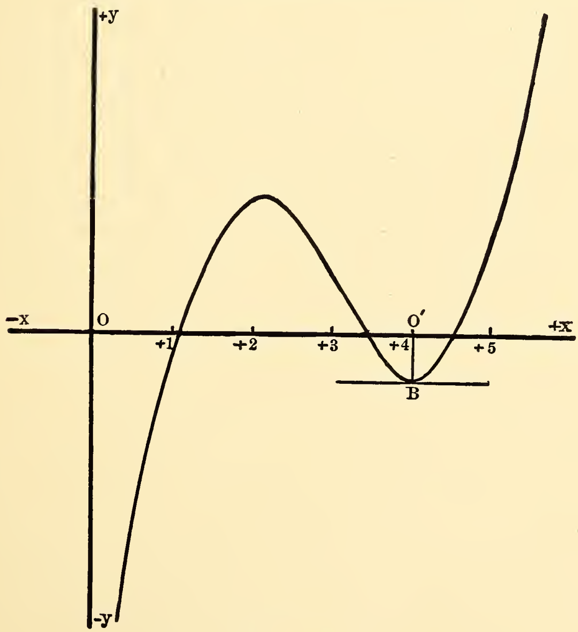

FIG. 35.

The plot of this equation shows that the integral parts of the three roots are respectively

1, 3, and 4.

In approximating to the third root we have

$$
\begin{array}{rrrrl}
1 & -9 & +24 & -17 & \underline{|4} \\
 & 4 & -20 & +16 & \\
\hline
1 & -5 & +4 & -1 & \text{First dividend.} \\
 & 4 & -4 & & \\
\cline{1-3}
1 & -1 & 0 & \text{First divisor.} & \\
 & 4 & & & \\
\cline{1-2}
1 & +3 & & &
\end{array}
$$

The divisor, or the value of the derivative when $x = 4$, being o shows that the tangent is parallel to the *x-axis*. This is a case in which we must resort to the method of trial of Art. 68. We find by this method that the curve crosses the *x-axis* between .5 and .6 beyond the point O'. Forming the coefficients of the new equation, we find no further difficulty.

73. SECOND CASE. *Where the divisor and dividend have the same sign.*

Take the equation

$$x^3 - 9x^2 + 23x - 14 = 0.$$

Let us approximate to the third root, the integral part of which is 4.

$$
\begin{array}{rrrrl}
1 & -9 & +23 & -14 & \underline{|4} \\
 & 4 & -20 & 12 & \\
\hline
1 & -5 & +3 & -2 & \\
 & 4 & -4 & & \\
\cline{1-3}
1 & -1 & -1 & & \\
 & 4 & & & \\
\cline{1-2}
1 & +3 & & &
\end{array}
$$

Here the tangent to the curve at the point B, or where $x = 4$, as will appear from the sign of the divisor, lies in a direction contrary to the direction of the curve at the root in question. (Fig. 36.)

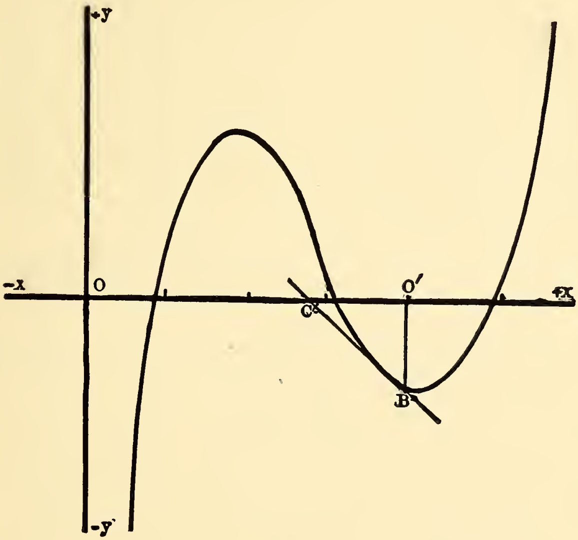

FIG. 36.

74. THIRD CASE. *Where the divisor and dividend have opposite signs, but the quotient figure is greater than the next figure of the root.* This is illustrated in Case 3 of the next article. Here, again, the next figure of the root must be found by trial.

In each of the above examples, the succeeding approximations will be obtained by the usual division.

There are cases, however, where the roots are so near

together that more than one decimal place must be found by trial.

The Four Positions of the Tangent.

75. The following example illustrates the different cases that may arise in the solution of an equation by Horner's method, with the corresponding positions of the tangent relative to the curve and the axis of x, when the division gives reliable information in regard to the root figure.

Take the equation

$$x^4 - 15x^3 + 77x^2 - 153x + 88 = 0.$$

On the opposite page is the plot.

First Case.

1	− 15	+ 77	− 153	+ 88	|.94
	.9	− 12.69	+ 57.879	+ 85.6087	
1	− 14.1	+ 64.31	− 95.121	+ 2.3911	First dividend.
	.9	− 11.88	47.187		
1	− 13.2	+ 52.43	− 47.934	First divisor.	
	.9	− 11.07			
1	− 12.3	+ 41.36			
	.9				
1	− 11.4				

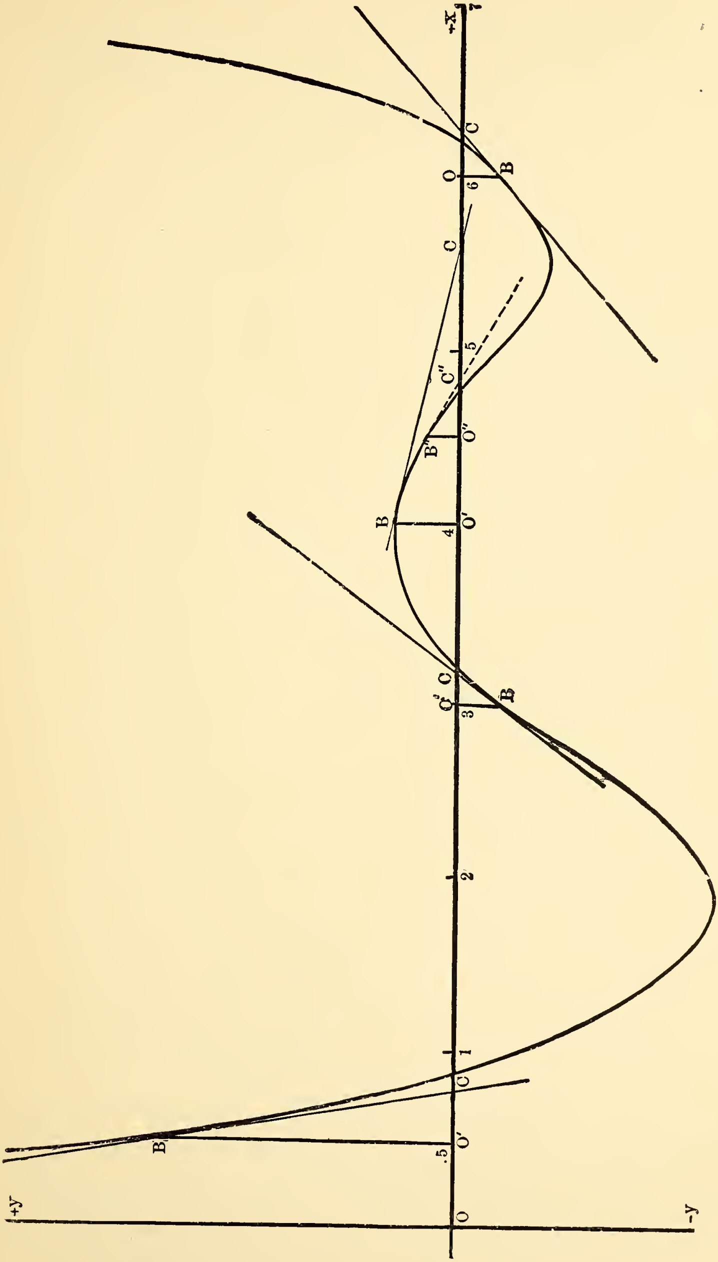

Fig 37.

Second Case.

1	− 15	+ 77	− 153	+ 88	3.1
	3	− 36	123	− 90	
1	− 12	+ 41	− 30	− 2	First dividend.
	3	− 27	+ 42		
1	− 9	+ 14	+ 12	First divisor.	
	3	− 18			
1	− 6	− 4			
	3				
1	− 3				

Third Case.

1	− 15	+ 77	− 153	+ 88	4
	4	− 44	+ 132	− 84	
1	− 11	+ 33	− 21	+ 4	First dividend.
	4	− 28	20		
1	− 7	+ 5	− 1	First divisor.	
	4	− 12			
1	− 3	− 7			
	4				
1	+ 1				

Fourth Case.

1	-15	$+77$	-153	$+88$	$\lvert\ \underline{6.1}$
	6	-54	$+138$	-90	
1	-9	$+23$	-15	-2 First dividend.	
	6	-18	$+30$		
1	-3	$+5$	$+15$ First divisor.		
	6	18			
1	$+3$	$+23$			
	6				
1	$+9$				

In the first case the dividend is positive and the tangent crosses before the curve.

In the second case the dividend is negative and the tangent crosses before the curve.

In the third case the dividend is positive and the tangent crosses beyond the curve.

In the fourth case the dividend is negative and the tangent crosses beyond the curve.

Abbreviation of the Work.

76. Find the value to six decimal places of one root of the equation

$$x^4 - 8x^3 + 14x^2 + 4x - 8 = 0,$$

the integral part of the root bein[illegible]

A	B		C		D		V		
1	− 8		+ 14		+ 4		− 8		5.2360679
	− 3		− 1		− 1		− 5		
	+ 2		+ 9		+ 44 = 1st divisor.		− 13 = 1st dividend.		
	7		44		53.288		10.6576		
	12.2		46.44		63.072 = 2d divisor.		− 2.3424 = 2d dividend.		
	12.4		48.92		64.626	747	1.9388	0241	
	12.6		51.44		66.193	068 = 3d div'r.	− .4035	9759 = 3d divid'd.	
	12.8	3	51.82	49	66.509	117736	.3990	54706416	
	12.8	6	52.21	07	66.825	633024 = 4th d'r.	− .0045	42883584 = 4th d'd.	
	12.8	9	52.59	74					
	12.9	26	52.67	4956					
	12.9	32	52.75	2548					

The above calculation is exhibited in an abbreviated form. Thus, in making the substitution, we multiply A by 5, add the product to B, and set down simply the result, which is -3. We do the same in each of the columns, thus dispensing with half the number of lines employed in preceding examples.

Again, when we wish to obtain a root correct to a limited number of places, we may omit much of the labor by cutting off all figures beyond a certain decimal; that is, we need set down none of those figures on the right of each of the vertical lines drawn in the several columns.

It will also be seen that instead of bringing down the results after each transformation we may leave them as above, and that in the last division we may carry out the quotient to several figures without committing any error.

The student may plot the above example and find the remaining positive roots.

Ans. $\begin{cases} .7639320 \\ 2.7320508 \end{cases}$

77. *Application of Horner's Method to the Extraction of the Roots of Numbers.* The nth root of any number a may be found by Horner's method by solving the equation

$$x^n = a.$$

Example. Let it be required to extract the cube root of 16 to five decimal places.

1	+ 0	+ 0	− 16	2.51984 +
1	2	4	− 8	
1	4	12	− 0.375	
1	6.5	15.25	− 0.187	
1	7.0	18.75	− 0.016	
1	7.5	18.83	− 0.001	
1	7.6	18.90		
1	7.6	18.97		
1	7.6	19.04		
1	7.6	19.05		

Examples.

1. Extract the cube root of 48228544. Ans. 364.
2. Required the fourth root of 18339659776.
3. Extract the fourth root of .2975 to four decimal places.
4. Extract the fifth root of 9 to three decimal places.

Example. Let it be required to extract the fifth root of

2628.6674882643.

					| 4.83
1	+ 0	+ 0	+ 0	+ 0	− 2628.6674882643
	4	16	64	256	− 1604.6674882643
	8	48	256	1280	− 80.6278082643
	12	96	640	1905.0496	0.0000000000
	16	160	781.312	2654.2080	
	20	176.64	936.448	2687.59360881	
	20.8	193.92	1105.990		
	21.6	211.84	1112.853627		
	22.4	230.40			
	23.2	231.1209			
	24.0				
	24.03				

Symmetry. Negative Roots by Horner's Method.

78. *Symmetry with respect to an Axis.* Two points are symmetrical with respect to an axis when this axis bisects at right angles the line joining these points.

Two figures, as for example two curves, are symmetrical with respect to an axis when every point in one has its symmetrical point in the other.

If in any equation we substitute $-x$ for x, y remaining unchanged, we form an equation whose curve is symmetrical to the curve of the original equation with respect to its *y-axis*.

For any point as P in the curve AB has for its symmetrical point P' on the curve $A'B'$. This substitution, it will be seen, will change the signs of the terms containing odd powers of x. Changing the signs of the terms having odd powers may therefore be interpreted as revolving the curve over the *y-axis* through two right angles, and it will be seen

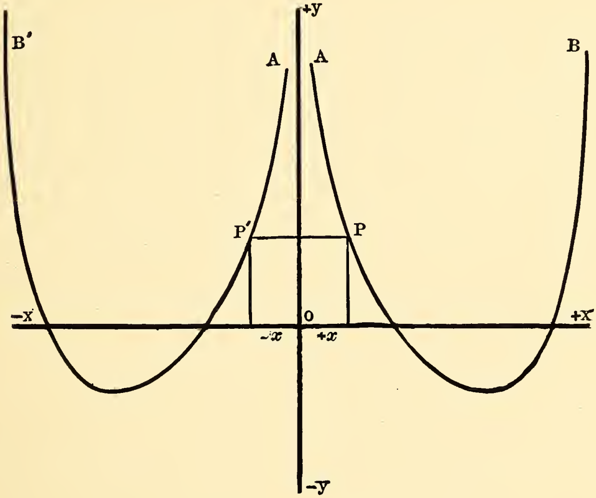

FIG. 38.

from the above figure that this will change the signs of all the roots of the equation.

79. *Symmetry with respect to a Centre.* Two points are symmetrical with respect to a centre when the line joining these points is bisected by this centre.

Two figures, as for example two curves, are symmetrical with respect to a centre when every point of one has its symmetrical point in the other.

If in any equation the signs of the terms having even exponents be changed, an equation is formed whose curve is symmetrical to the curve of the original equation with respect to the origin as a centre.

First. We change the signs of *all* the terms of the equa-

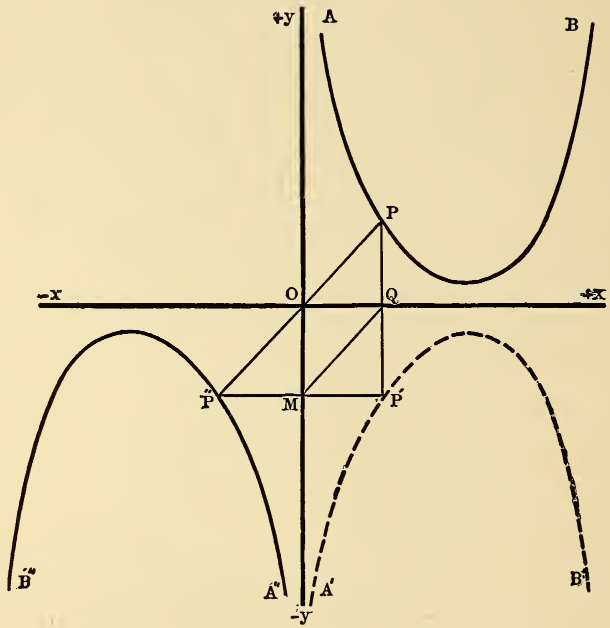

FIG. 39.

tion. This is the same as putting $-y$ for y in the equation for plotting, x remaining unchanged; and this transformed equation gives the curve $A'B'$, which is symmetrical to AB with respect to the *x-axis*, by the principle employed in the last article.

Second. We change the signs of the *odd* terms in this last

equation, and this gives a curve $A''B''$ symmetrical to $A'B'$ with respect to the *y-axis*, by the last article.

The resulting equation will differ from the original only in the signs of the terms involving even powers.

AB is the curve of the original equation, and $A''B''$ of the resulting equation. The curve AB is symmetrical to $A''B''$ with respect to O as a centre. For, since $PQ = P'Q = OM$ and $OQ = P''M$, it follows that $PO = P''O$. Now $P''O$ and OP form one and the same straight line, because both pass through O and are parallel to QM.

And since the point P is *any* point of the curve AB, it follows that the curves AB and $A''B''$ are symmetrical with respect to the origin O as a centre.

80. From these considerations it follows that changing the signs of the alternate terms of an equation, beginning with the second term, revolves its curve through two right angles over the *y-axis* if the equation be of an even degree, and about the origin as a centre if the equation be one of an odd degree; and in both cases the signs of all the roots of the equation are changed, while the numerical values of these roots remain unchanged.

Examples.

1. The roots of the equation

$$x^3 - 2x^2 - 5x + 6 = 0$$

are $\qquad 1, 3, \text{ and } -2;$

what are the roots of the equation

$$x^3 + 2x^2 - 5x - 6 = 0?$$

2. The roots of the equation

$$x^4 + 6x^3 + 5x^2 - 2x - 10 = 0$$

are $+1, -5, -1 + \sqrt{-1}$, and $-1 - \sqrt{-1}$;

what is the equation whose roots are

$$-1, +5, 1 + \sqrt{-1}, 1 - \sqrt{-1}?$$

3. Find by Horner's method the roots of the equation

$$x^3 + 9x^2 + 24x + 17 = 0.$$

4. Find by Horner's method the roots of the equation

$$x^4 - 12x^2 - 12x = 3.$$

5. Find the four roots of the equation

$$x^4 + 8x^3 + 14x^2 - 4x = 8.$$

CHAPTER V.

Discussion of Equations of the Second, Third, and Fourth Degrees.

Preliminary Principles.

81. *Moving the Origin Up and Down.* If the equations

$$y = x^3 - 4x^2 - 11x + 10$$

and $$y' = x^3 - 4x^2 - 11x + 30$$

be plotted on the same axes, it will be found that any point

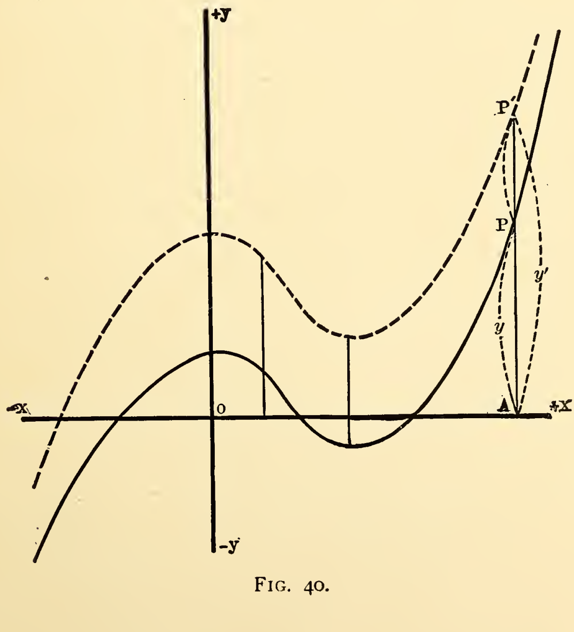

Fig. 40.

of the curve of the second equation, as P', is 20 units *above* a corresponding point, P, of the first. (See Art. 38.)

That is, if y' is $P'A$ and y is PA, PP' being 20,

$$y' = y + 20.$$

If now we draw a second axis of x through the point O'

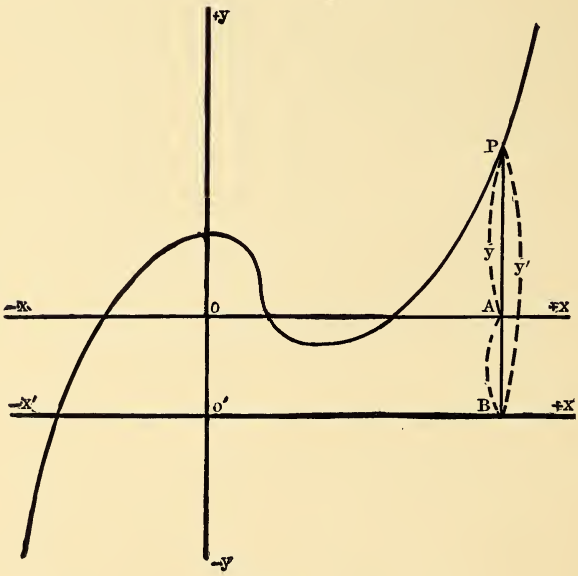

FIG. 41.

20 units below the point O, and plot the second equation from this new origin, the second curve will coincide throughout with the first because every point is brought down 20 units.

In this case AB is 20, PA is y, and PB is y'. Hence

$$y' = y + 20,$$

as before.

Hence the change in the plot caused by the change in the absolute term of an equation may be expressed by the following

RULE. *If the absolute term of an equation be increased by a units, the origin must be moved downward on the y-axis a units; and if the absolute term be diminished by a, the origin must be moved upward a units.* Conversely, *if the origin be moved along the y-axis $\mp$ a units, the absolute term of the equation must be changed by $\pm$ a units.*

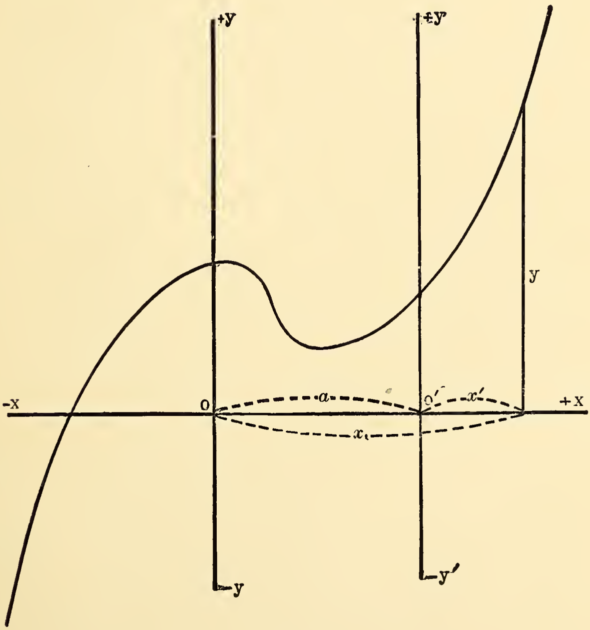

FIG. 42.

82. *Moving the Origin Right and Left.* We have seen in Art. 64 that by substituting in an equation $x' + a$ for x we

move the origin on the plot a units to the right without changing the form of the curve. By substituting $x' - a$ for x it is also evident that the origin will be moved a units to the left. (Fig. 42.)

83. In the discussion of equations we have an important application of the principle of moving the origin right and left, in the *removal of the second term of an equation.*

In the substitution of $x' + a$ for x in an equation of the nth degree, the second term of the resulting equation, as we have seen, is formed by adding a n times, or by adding na to the second term of the original equation. If now we call the second term of the original equation B, we can make the second term of the resulting equation vanish by putting

$$na = -B \qquad \text{or} \qquad a = -\frac{B}{n}.$$

Hence, to transform an equation into another whose second term is wanting, *substitute for x, x' minus the co efficient of the second term divided by the highest exponent of x.*

According to Art. 27, when the second term of an equation is wanting the sum of the positive roots is numerically equal to the sum of the negative roots.

84. *Removing the Second Term of a Quadratic Equation.* In the equation

$$x^2 - 6x + 5 = 0$$

we may remove the second term by putting

$$x = x' - \left(\frac{-6}{2}\right) \qquad \text{or} \qquad x = x' + 3.$$

$$\begin{array}{rrr|l} 1 & -6 & +5 & 3 \\ & 3 & -9 & \\ \hline 1 & -3 & -4 & \\ & 3 & & \\ \hline 1 & 0 & & \end{array}$$

Or $$x'^2 - 4 = 0.$$

Whence

$$x' = \pm 2 \quad \text{or} \quad x = \pm 2 + 3 = +1 \text{ or } +5.$$

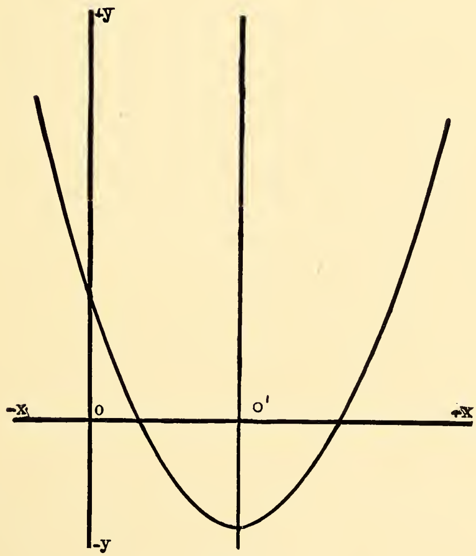

FIG. 43.

Fig. 43 is the plot of the equation

$$y = x^2 - 6x + 5.$$

Removing the second term moves the origin three units to the right, or to O', and by Art. 78 the curve is symmetrical

with respect to the new *y-axis*. It will be seen that this gives an interpretation of the process of solving a quadratic equation by the ordinary method of completing the square.

Examples.

1. Transform $x^4 - 12x^3 + 6x^2 - 7x + 4 = 0$ into an equation whose second term is wanting, plot the equation, and indicate the new origin.

2. Transform $x^3 - 6x^2 + 8x - 2 = 0$ into an equation whose second term is wanting.

3. Transform $x^5 - 15x^4 + 12x^3 - 20x^2 + 14x - 25 = 0$ into an equation whose second term is wanting.

4. Solve the equation $x^2 - 3x - 10 = 0$ by removing the second term. Plot the equation, indicate the new origin, and measure the roots on the plot.

85. *Changing the Scale of the Curve of an Equation.* If we put ax' for x in any equation, we form a new equation in which every value of x' is the $\frac{1}{a}$ part of the corresponding value of x in the original equation.

In the figure the curve AB is the plot of the equation

$$y = x^3 - 4x.$$

If now we put $x = 2x'$, we obtain the equation

$$y = 8x^{3\prime} - 8x'.$$

the plot of which is $A'B'$.

In this last curve the x' of each point is $\frac{1}{2}$ of the x for the corresponding point on AB.

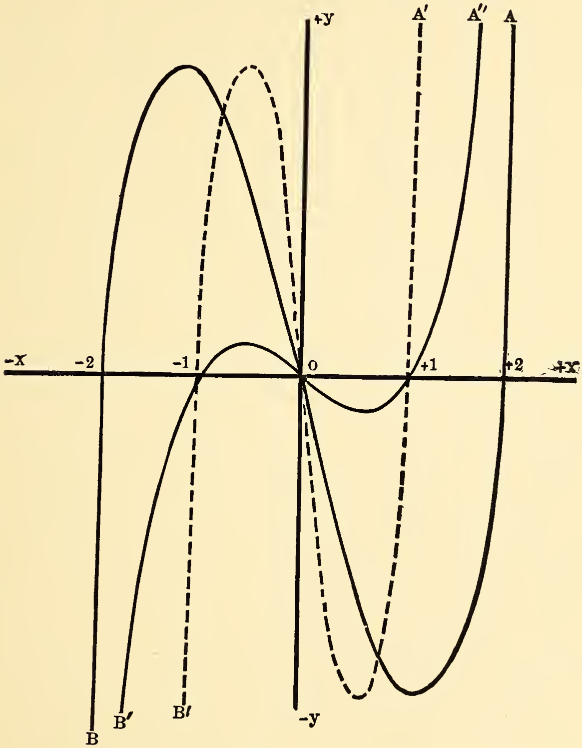

FIG. 44.

Dividing the last equation by 8 gives

$$\frac{y}{8} = x^{3'} - x' \qquad \text{or} \qquad y' = x'^3 - x';$$

by putting $y' = \frac{y}{8}$ or $y = 8y'$, and plotting, we obtain the curve $A''B''$.

In this article we have an interpretation of the process employed in transforming an equation having fractional coef-

ficients into another which has all its coefficients integers and the coefficient of its first term unity.

Example. Transform the equation

$$x^3 - \frac{3x^2}{2} + \frac{5x}{4} - \frac{2}{9} = 0$$

into another whose coefficients are integers and that of the first term unity.

Clearing of fractions, we have

$$36x^3 - 54x^2 + 45x - 8 = 0.$$

Substituting $\frac{x'}{6}$ for x, the equation becomes

$$\frac{x'^3}{6} - \frac{9x'^2}{6} + \frac{45x'}{6} - 8 = 0,$$

or
$$x'^3 - 9x'^2 + 45x' - 48 = 0.$$

The student can draw the plot and interpret the various steps of the process.

Examples. Transform the following equations into others whose coefficients shall be integers and that of the first term in each shall be unity:

(1) $x^3 - 2x^2 + \frac{x}{4} + \frac{1}{9} = 0.$

(2) $x^3 - \frac{x^2}{3} - \frac{x}{4} - 2 = 0.$

(3) $x^3 + 2\frac{1}{6}x^2 + \frac{x}{6} - \frac{1}{3} = 0.$

(4) $x^4 + 4\frac{1}{2}x^2 + 8x + 2\frac{1}{8} = 0.$

(5) $x^3 - \frac{14x^2}{3} - 7x - \frac{10}{3} = 0.$

The Interpretation of Imaginary Quantities.

86. In Art. 40 we saw that when an equation of the nth degree had imaginary roots—*i.e.*, roots in which $\sqrt{-1}$, $\sqrt[4]{-1}$, etc.—occurred, the presence of these imaginaries was indicated by the curve crossing the *x-axis less than n times.* In this chapter we shall show how the imaginary branches of a curve may be plotted so that the imaginary roots of the equations which we are considering may be measured on the plot.

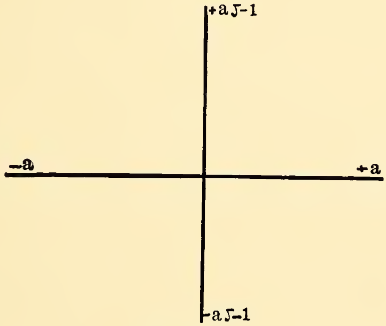

FIG. 45.

87. To distinguish imaginary from real values on the plot, it has been agreed to lay off the imaginary values on lines at right angles to those on which real values are laid off. That is, if we have the following values of x, viz.,

$$x = +a,$$

$$x = -a,$$

$$x = +a\sqrt{-1},$$

$$x = -a\sqrt{-1},$$

we may lay them off as in the figure. But in order that the imaginary values of x shall not interfere with values of y, it will be necessary to lay them off on lines perpendicular to the plane of the x and *y-axes*. The method of drawing best adapted to represent all these lines in one figure is the

Isometric Projection.

88. Inscribe a regular hexagon in a circle by laying off the radius six times in succession on the circumference;

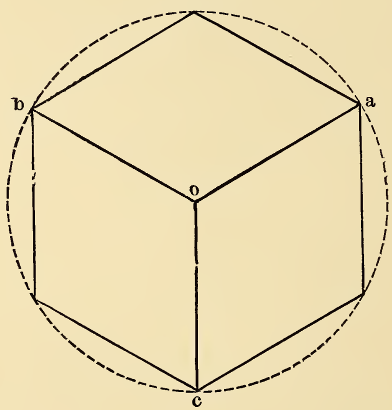

FIG. 46.

then join the alternate vertices as in Fig. 46. This is called the isometric projection of the cube, and represents the cube as it would appear if it were placed so that a line drawn through two of its diagonally opposite corners should be perpendicular to the plane of the paper. From the con-

struction of the figure it is evident that $aob = boc = aoc = 120°$ and $ao = bo = co$.

Application to Curve Drawing.

89. Since $ao = bo = co$, the edges of the cube are all represented on the same scale, and it follows that *all distances measured on lines parallel to the edges of a cube may*

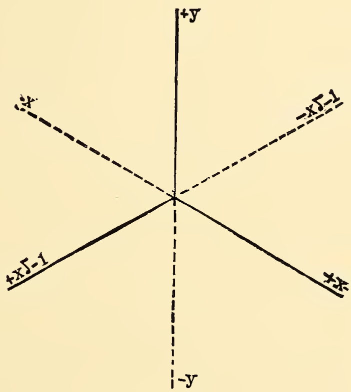

FIG. 47.

be drawn on a uniform scale. Hence oc, ob, and ao may represent respectively the directions of the y, x, and $x\sqrt{-1}$ axes or lines of reference, and all values of y, x, and $x\sqrt{-1}$ may be correctly represented on a uniform scale on lines parallel to these axes. Fig. 47 shows the axes with their appropriate signs.

90. The isometric projection furnishes the only method

of plotting the real and imaginary values on the same figure by a uniform system of measurements.

91. Paper ruled as in Fig. 48 is convenient for Isometric Drawing.

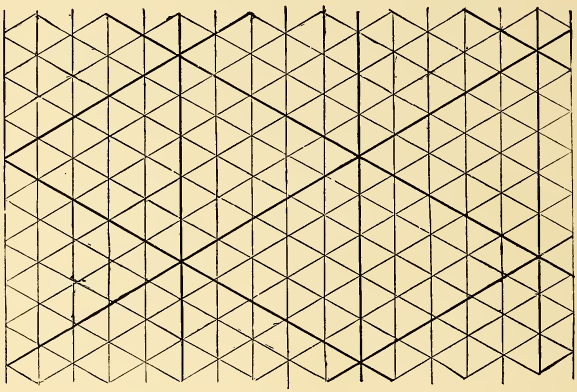

FIG. 48.

92. *To Plot both the Real and Imaginary Curves of the Equation*

$$y = x^2 - 6x + 8. \qquad \text{(Fig. 49)}$$

The table of values for the real curve may be made in the usual manner, and these values may be laid off by measuring each value of x on the *x-axis*, and then from the points so determined measuring the corresponding values of y on lines parallel to the *y-axis*.

93. *To Draw the Imaginary Part of the Curve.* Instead of assuming values of x and computing the corresponding

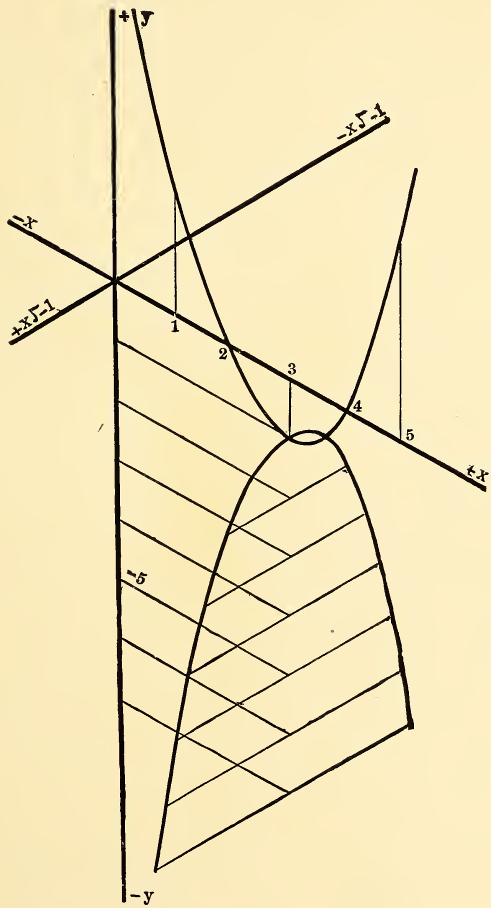

FIG. 49.

values of y, we assume in succession values of y differing from each other by unity, and, solving the equation, compute the corresponding values of x.

y	x
-1	3
-2	$3 \pm 1\sqrt{-1}$
-3	$3 \pm 1.4\sqrt{-1}$
-4	$3 \pm 1.7\sqrt{-1}$
-5	$3 \pm 2\sqrt{-1}$
-6	$3 \pm 2.2\sqrt{-1}$

In this way we obtain the table of values for the imaginary curve.

Measure on the *y-axis* the successive values of y in the table, and from each point so determined measure the real part of x on a line parallel to the *x-axis*, and from thence on a line parallel to the $x\sqrt{-1}$ axis lay off the imaginary part of x. The curve traced through the points so determined will be the imaginary part of the plot.

94. *The $\sqrt{-1}$ is a symbol of direction.* In plotting, the student must always regard $\sqrt{-1}$ as merely a symbol like $+$ or $-$. It indicates that the number to which it is affixed is to be plotted in a particular direction. This interpretation may be shown to be consistent with the ordinary interpretation given to the signs $+$ and $-$ when we consider $\sqrt{-1}$ as a factor of a number.

Suppose we take any number, as $+a$. This we lay off by measuring a units to the right from 0. If now we multiply $+a$ by $\sqrt{-1}$, we get $+a\sqrt{-1}$. Lay this off by measuring a units from 0 at right angles to the line just measured. If we multiply $a\sqrt{-1}$ by $\sqrt{-1}$, we have, by the rules for multiplying imaginaries, $-a$; and if we meas-

ure a units from o along a line at right angles to the line on which $a\sqrt{-1}$ was measured, we find that it coincides with the line $-a$ laid off from o to the left. If we use the $\sqrt{-1}$ as a multiplier twice more the results are respectively

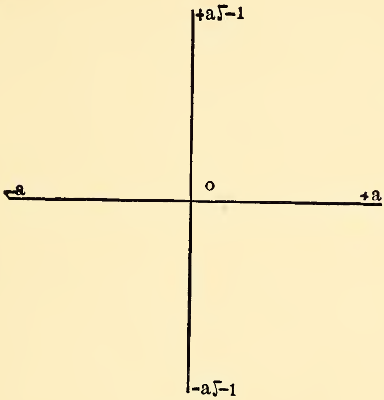

FIG. 50.

$-a\sqrt{-1}$, and $+a$ which are laid off as in the figure. The The result of these operations is sometimes expressed by saying that the $\sqrt{-1}$ *used as a multiplier turns a line through one right angle.*

Discussion of Equations.

95. *Equations of the Second Degree.* Every equation of the second degree of the form

$$y = x^2 + ax + b,$$

where a and b are any real numbers, positive or negative, gives when plotted a curve of the same form as the curve of the equation

$$y = x^2.$$

For the second term of the second member of the first equation may be made to vanish, according to Art. 84, by moving the origin $\frac{a}{2}$ units to the right or left according as a is a minus or plus number. This reduces the equation to the form

$$y = x'^2 + c.$$

Also, the absolute term c in this last equation may be made to vanish by moving the origin up or down according as c is plus or minus. In both of these transformations the form of the curve remains the same.

Therefore upon the curve of the equation

$$y = x'^2$$

we may find by measurement the roots of any equation

$$x^2 + ax + b = 0,$$

whether these roots be real or imaginary.

Thus in the equation

$$x^2 - 2x + 5 = 0$$

we may remove the second term by substituting

$$x = x' + 1,$$

according to Art. 83.

Thus:

$$\begin{array}{ccc|l} 1 & -2 & +5 & \underline{1} \\ & +1 & -1 & \\ \hline 1 & -1 & +4 & \\ & +1 & & \\ \cline{2-2} 1 & 0 & & \end{array}$$

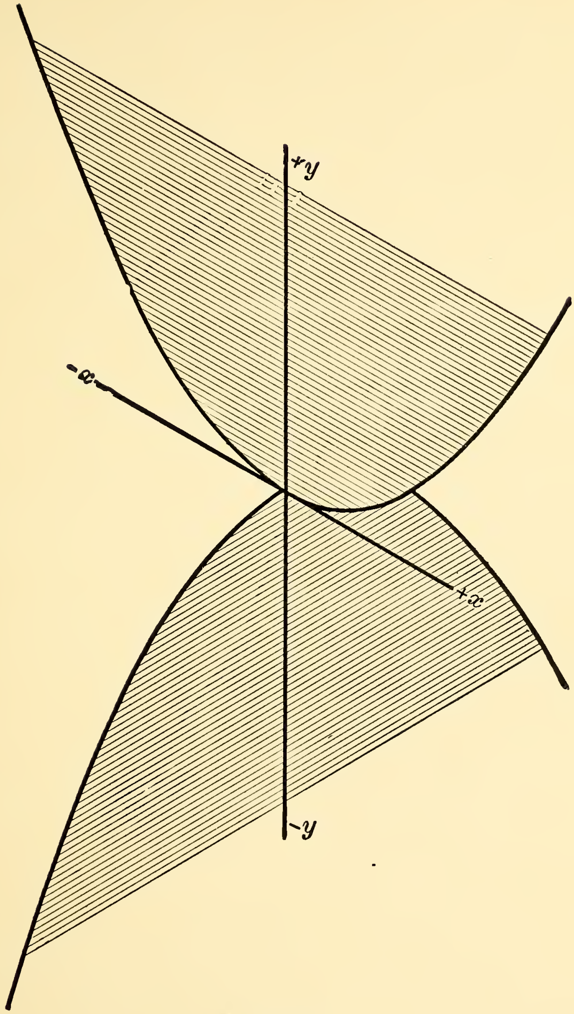

FIG. 51.

Whence $$x'^2 + 4 = 0.$$

If now in the plot of the equation

$$y = x'^2 \qquad \text{(Fig. 51)}$$

the origin be moved downward four units, we shall have the plot of

$$y = x'^2 + 4,$$

upon which the roots of

$$x'^2 + 4 = 0$$

are found by measurement to be $\pm 2\sqrt{-1}$.

And since these roots are the values of x', we may find x from the equation

$$x = x' + 1 \quad \text{or} \quad x = 1 \pm 2\sqrt{-1}.$$

The algebraic operations which we have performed here are analogous to the operations by which quadratic equations are ordinarily solved.

96. *Equations of the Third Degree.* Every equation of the third degree of the form

$$y = x^3 + ax^2 + bx + c,$$

where a, b, and c are positive or negative numbers, gives when plotted a curve which belongs to one of three distinct types.

The second term of the right-hand member of this equation may be made to disappear by substituting $x' - \frac{a}{3}$ for x. The equation will then become

$$y = x'^3 + nx' + m,$$

where n and m are either positive or negative.

This transformation does not change the form of the curve,

Art. 82. Again, the form of the curve is independent of the absolute term m, and therefore this term may be neglected in plotting.

Hence the curve of every equation of the third degree depends for its form entirely upon the coefficient n in this transformed equation. Giving to the x' term the proper sign of n, we have for the three forms of the equation of the third degree, omitting the primes,

$$y = x^3 - nx,$$
$$y = x^3 + nx,$$
$$y = x^3,$$

each one of which gives a curve of a distinct type. The curve of the first of these equations has two distinct elbows; that of the second has none; and the third equation, where n is zero, gives the limiting case of the other two forms.

97. *Plotting of the Real and Imaginary Branches of the Curves of the above Equations.* Assume a series of real and positive values for x, and compute the corresponding values of y by the ordinary method. The same series of negative values of x give values of y unchanged numerically, but with opposite signs. We lay off this table of values in isometric projection and draw the real part of the curve.

To illustrate the method of plotting the imaginary part of the curve we take the equation

$$y = x^3 - x,$$

which is an example of the first form in the above group, where $n = 1$.

The derivative of the right-hand member of this equation is

$$3x^2 - 1.$$

Putting this equal to zero and solving gives

$$x = \pm .577,$$

and substituting these values in

$$y = x^3 - x,$$

we obtain $$y = \pm .385.$$

These values of y determine the limits of the imaginary branches; that is, the imaginary branches lie entirely outside the limits

$$y = + .385 \quad \text{and} \quad y = - .385.$$

For a point on the real branch of the curve corresponding to a value of y greater than $+ .385$ let us suppose $x = a$. Then for this value of y there will also be two imaginary values of x of the form

$$x = - b + c\sqrt{-1} \quad \text{and} \quad x = - b - c\sqrt{-1}.$$

Then the product of the three factors formed from these values of x, viz.,

$$(x - a), \quad (x + b - c\sqrt{-1}), \quad \text{and} \quad (x + b + c\sqrt{-1})$$

put equal to zero, or

$$x^3 + (2b - a)x^2 + (b^2 + c^2 - 2ab)x - a(b^2 + c^2) = 0, \qquad (1)$$

will correspond to the equation

$$y = x^3 - x,$$

when the origin is moved upward a distance equal to the assumed value of y.

Equation (1) may be written

$$a(b^2 + c^2) = x^3 + (2b - a)x^2 + (b^2 + c^2 - 2ab)x.$$

Comparing it member by member and term by term with $y = x^3 - x$, we have

$$y = a(b^2 + c^2),$$

$$2b - a = 0,$$

and $$b^2 + c^2 - 2ab = -1;$$

whence $$b = \frac{a}{2} \quad \text{and} \quad c = \pm\sqrt{\frac{3a^2}{4} - 1}.$$

Assume $a = 1.2$; then

$$b = .6 \quad \text{and} \quad c = \pm .283.$$

Hence $$y = + .528,$$

and $$x = 1.2,$$

$$x = - .6 + .283\sqrt{-1},$$

$$x = - .6 - .283\sqrt{-1}.$$

Assuming the real part of the curve plotted as in the figure, we lay off the imaginaries found above as follows:

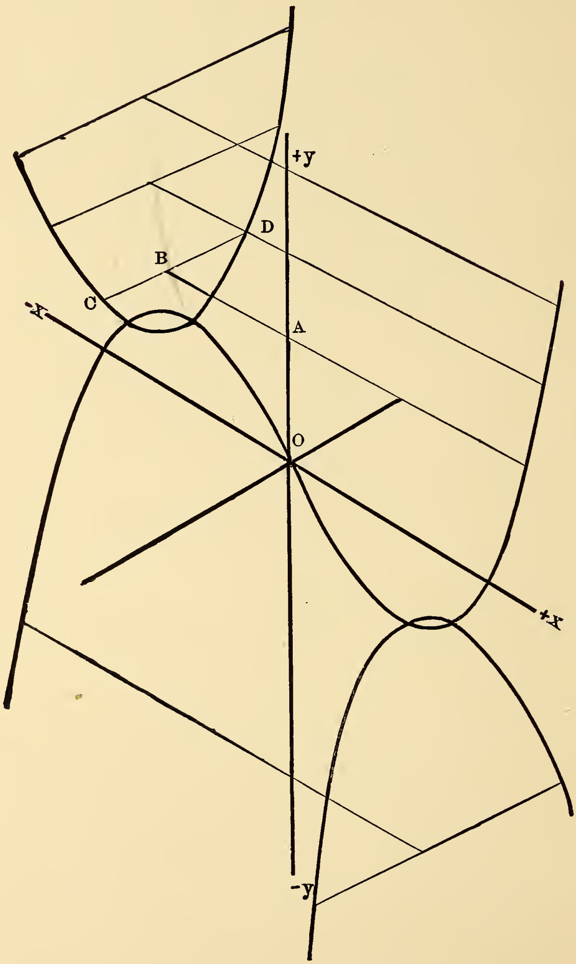

Fig. 52.

Measure from the origin upward on the *y-axis* .528, or OA,

thence to the left $-.6$, or AB, parallel to the *x-axis*, and from

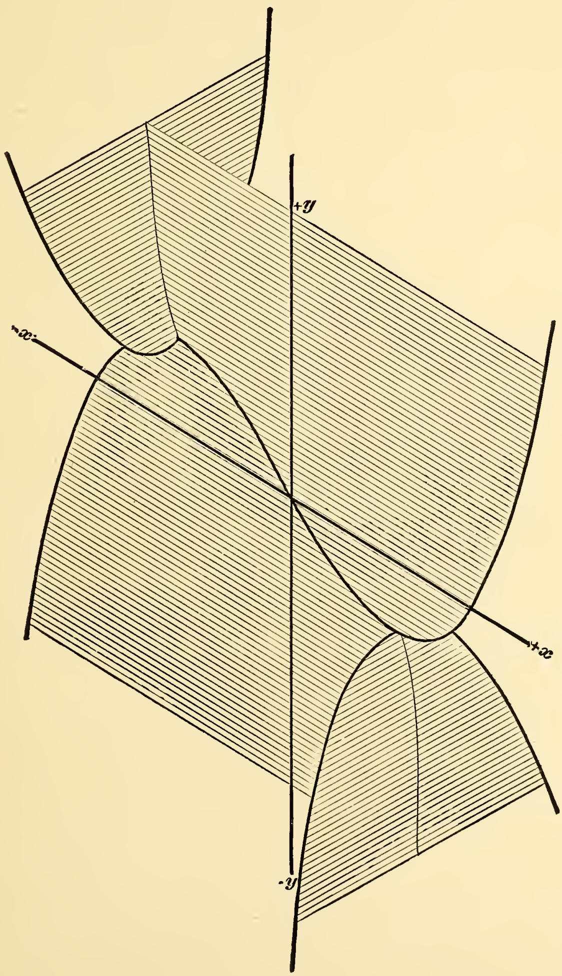

FIG. 53.

the point B so determined measure .283 in both directions,

viz., BC and BD, parallel to the $x\sqrt{-1}$ axis. In like manner any number of points may be determined for the imaginary curve:

When $a = 1.3$, $x = -.65 \pm .32\sqrt{-1}$;

When $a = 1.4$, $x = -.7 \pm .69\sqrt{-1}$;

etc. etc.

98. Fig. 53, page 111, is the plot of the equation

$$y = x^3 - x.$$

In the following curves the same scale is employed for x and y, the unit being the distance from the origin to the intersection of the curve with the *x-axis* in Fig. 53.

Example. Make a table of values for the real part of the above curve by assuming values of x differing from each other by $\frac{1}{10}$. Also a table of imaginary values assuming values of a at intervals of $\frac{1}{10}$.

Plot of the equation

$$y = x^3 - \tfrac{1}{10}x. \qquad \text{(Fig. 54.)}$$

As the coefficient of x is made smaller, the intersections of the curve with the *x-axis*, the elbows, and the imaginary branches approach each other.

Example. Find algebraically the limits between which no part of the imaginary branches lie, and make a table of values for plotting the imaginary branches of the curve.

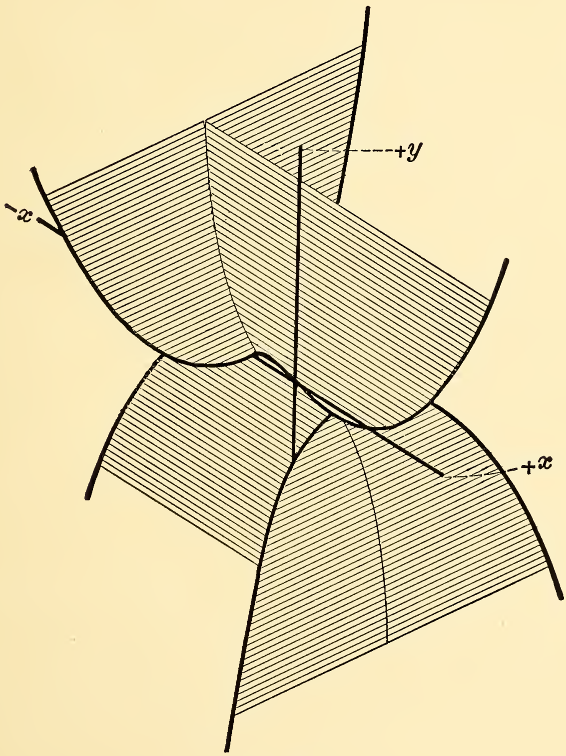

FIG. 54.

Plot of the equation

$$y = x^{1/3}. \qquad \text{(Fig. 55)}$$

In this plot it will be seen that the imaginary branches of the curve meet at the origin.

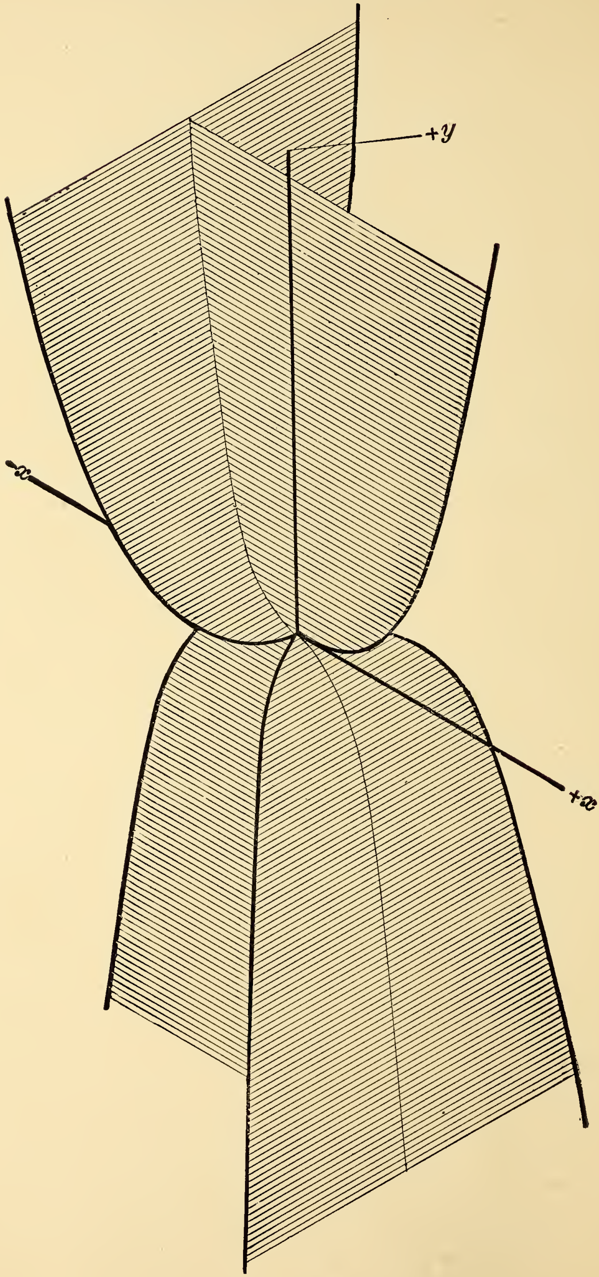

FIG. 55

Example. Compute a table of values for plotting the real and imaginary branches of the curve (Fig. 55) by assuming values of y at intervals of $\frac{1}{10}$.

Plot of the equation

$$y = x^3 + \tfrac{1}{10}x. \qquad \text{(Fig. 56.)}$$

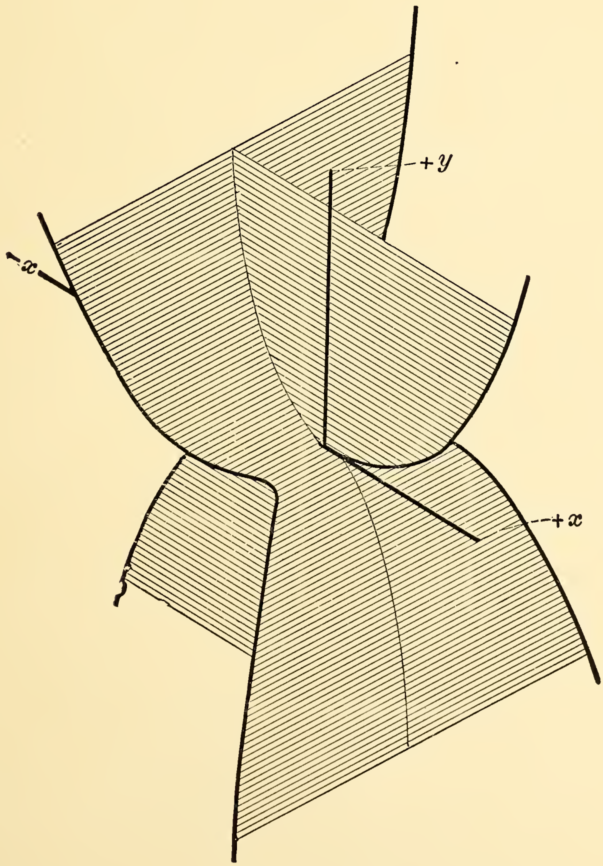

FIG. 56.

As soon as the coefficient of x becomes positive, the imagin-

ary branches begin to take shape as in Fig. 56, and their distance apart increases as the coefficient of x increases.

Problem. Plot the real and imaginary branches of the equation
$$y = x^3 + \tfrac{1}{100}x.$$

Fig. 57 is a plot of the following equation,

$$y = x^3 + x.$$ $y = -ix^3 + ix$

In this curve the distance from the origin to the imaginary curve measured on the $x\sqrt{-1}$ axis is unity in both directions, and it will be seen that the imaginary curves approach more and more nearly to straight lines as the coefficient of x increases.

Example. Reduce the equation $y = x^3 - 6x^2 + 2x + 4$ to the form $y = x'^3 - nx' + m$ and plot it, showing both the new and the original axes.

99. *Graphic Solution of Cubic Equations of the form*

$$x^3 + ax^2 + bx + c = 0.$$

1st. Substitute $x = x' - \dfrac{a}{3}$. (Art. 84)

This reduces the equation to

$$x'^3 \pm nx' \pm m = 0;$$

2d. Substitute in this last equation

$$x' = n^{\frac{1}{2}}x''.$$

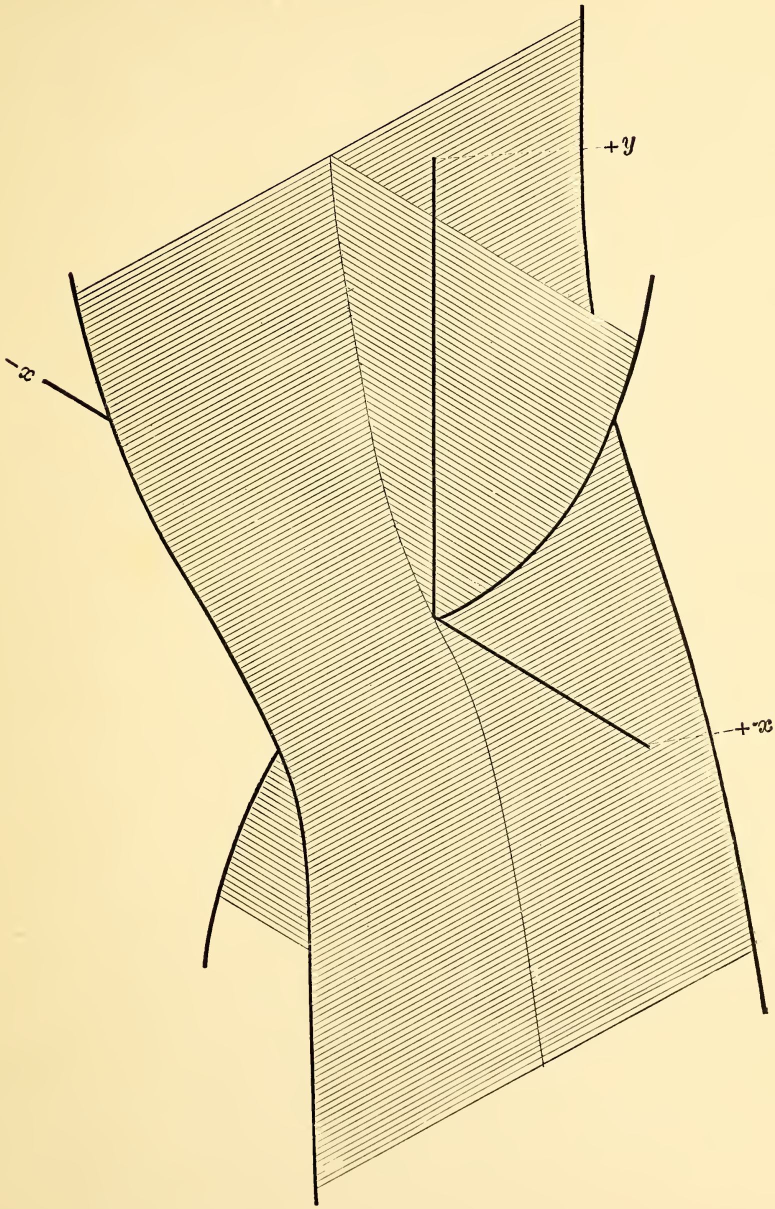

FIG. 57.

This reduces the equation to

$$x''^3 - x'' \pm \frac{m}{n^{\frac{3}{2}}} = 0 \text{ if the sign of the } x' \text{ term is } -;$$

$$x''^3 + x'' \pm \frac{m}{n^{\frac{3}{2}}} = 0 \text{ if the sign of the } x' \text{ term is } +;$$

$$x'^3 \pm m = 0 \text{ if } n \text{ is zero.}$$

The roots of any equation of the first of these three forms may be measured off on the plot of

$$y = x^3 - x$$

as follows: Find the value of $\frac{m}{n^{\frac{3}{2}}}$ to two or more decimal places. Measure this value from the origin on the *y-axis*, upward if its sign is minus, and downward if plus. Then the distances to the real or imaginary branches of the curve measured from this point, on lines parallel to the *x-axis*, and the $x\sqrt{-1}$ axis will give the roots of the equation

$$x''^3 - x'' \pm \frac{m}{n^{\frac{3}{2}}} = 0.$$

The accuracy of these roots will depend upon the accuracy of the plot.

Substitute these values of x'' so measured in the equation

$$x' = n^{\frac{1}{2}}x'';$$

and again, substitute the values of x' in the equation

$$x = x' - \frac{a}{3}.$$

The result will be the roots of the original equation.

The roots of the equations of the two remaining forms may be measured respectively on the plots of

$$y = x^3 + x$$

and

$$y = x^3.$$

The student is recommended to make plots of the three equations employed in this article, using two small squares of the ordinary cross-section paper to represent one tenth. Upon such plots well drawn very good approximations may be made with comparatively little labor.

Examples.

1. Find the roots of the equation

$$x^3 + 6x^2 - 4x = 2.$$

2. Find the roots of the equation

$$x^3 - 3x^2 + 2x = 1.$$

3. Find the roots of the equation

$$x^3 = -6.$$

4. Find the roots of the equation

$$x^3 - 4x + 2 = 0.$$

5. Find the roots of the equation

$$x^3 - 2x^2 = 6.$$

Equations of the Fourth Degree.

100. The general form of the equation of the fourth degree is

$$y = x^4 + Ax^3 + Bx^2 + Cx + D,$$

where A, B, etc., are real numbers, and may be either positive or negative. The second term of the second member may always be removed.

Again, the shape of the curve being independent of the absolute term, that term may be neglected in plotting, and we need only consider the equation

$$y = x^4 \pm ax^2 \pm bx.$$

101. *Symmetrical Forms of this Equation.* When $b = 0$ this equation reduces to

$$y = x^4 \pm ax^2,$$

which is symmetrical with respect to the *y-axis*, because when $-x$ is substituted for x the equation is unchanged. The form of the curve of this equation depends entirely upon a. When a is negative the real part of the curve has three distinct elbows; when a is positive there is one elbow; and when a is zero we have the limiting case of the other two forms. The three forms are, therefore,

$$y = x^4 - ax^2;$$
$$y = x^4 + ax^2;$$
$$y = x^4.$$

102. The real and imaginary branches of these curves may be plotted by assuming a series of values for y, and by the rule of quadratic equations computing the corresponding values of x. In the case, however, where the values of y make all the values of x imaginary, the formulas derived by

the following method may be employed. In the above equations the values of x are all imaginary for values of y below $-\frac{a^2}{4}$. It is evident also from the principle of symmetry that the four values of x corresponding to any value of y will be

$$x = m \pm p\sqrt{-1}$$

and

$$x = -m \pm p\sqrt{-1},$$

and the factors formed from these values, viz.,

$$(x - m + p\sqrt{-1}),$$
$$(x - m - p\sqrt{-1}),$$
$$(x + m - p\sqrt{-1}),$$

and

$$(x + m + p\sqrt{-1}),$$

when multiplied together and the result put equal to zero, that is,

$$x^4 + (2p^2 - 2m^2)x^2 + (m^2 + p^2)^2 = 0, \quad (1)$$

will correspond to the equation

$$y = x^4 \pm ax^2$$

when the origin in this last equation is moved along the *y-axis* a distance equal to the assumed value of y. Equation (1) may be written

$$-(m^2 + p^2)^2 = x^4 + (2p^2 - 2m^2)x^2.$$

Comparing this term by term with

$$y = x^4 \pm ax^2,$$

we have

$$y = -(m^2 + p^2)$$

and

$$\pm a = 2p^2 - 2m^2.$$

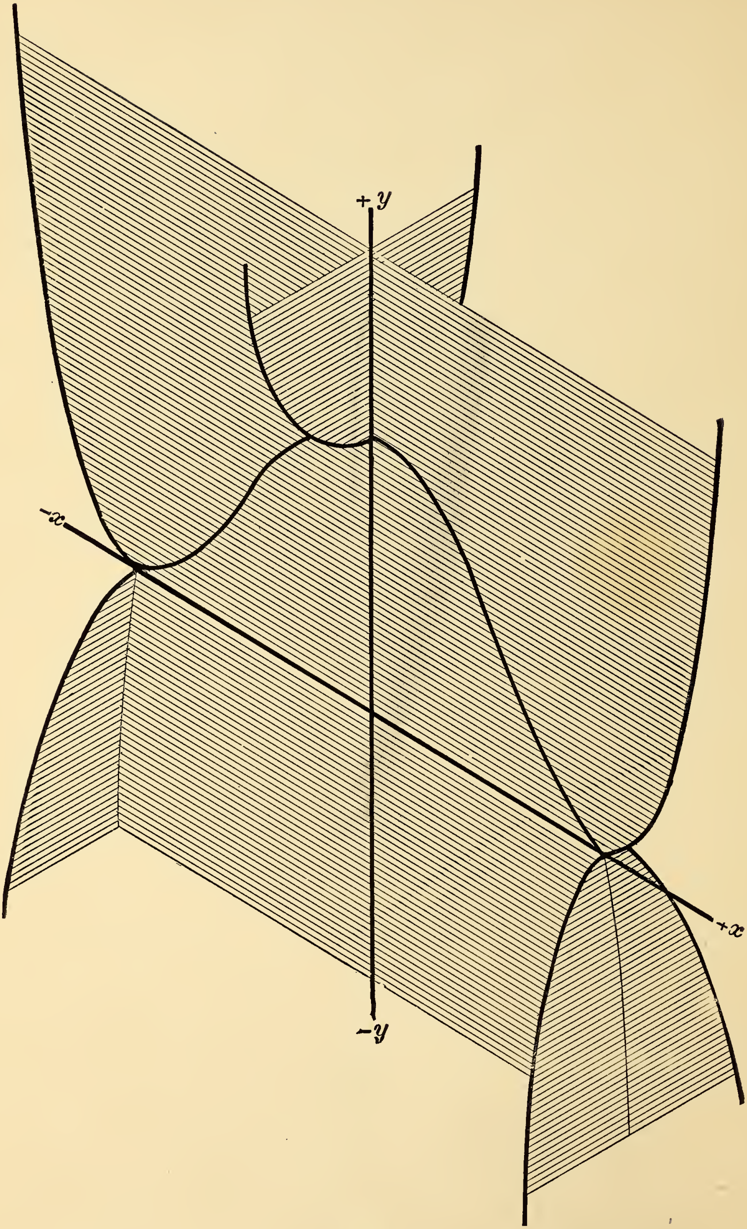

FIG. 58.

Assuming now successive values of m, the corresponding values of p and y are readily determined. These values may be laid off in the manner described in the last article.

As an example of the form

$$y = x^4 - ax^2,$$

we give the plot of the equation

$$y = x^4 - 2x^2 + 1.$$

Here the absolute term may be neglected in plotting, and afterwards the *x-axis* moved down one unit.

The limits of the imaginary branches, after the origin is moved as indicated above, are $y = 0$ and $y = +1$

Fig. 58 is the plot of the equation

$$y = x^4 - 2x^2 + 1.$$

This equation has two pairs of equal roots; viz., at $+1$ and -1.

103. As an example of the second form we will plot the equation

$$y = x^4 + 2x^2 + 1,$$

where the values for plotting are found in the same way as in the preceding equation.

Fig. 59 is the plot of this equation.

104. This equation furnishes an example of two pairs of equal imaginary roots. It will be seen that if this curve be

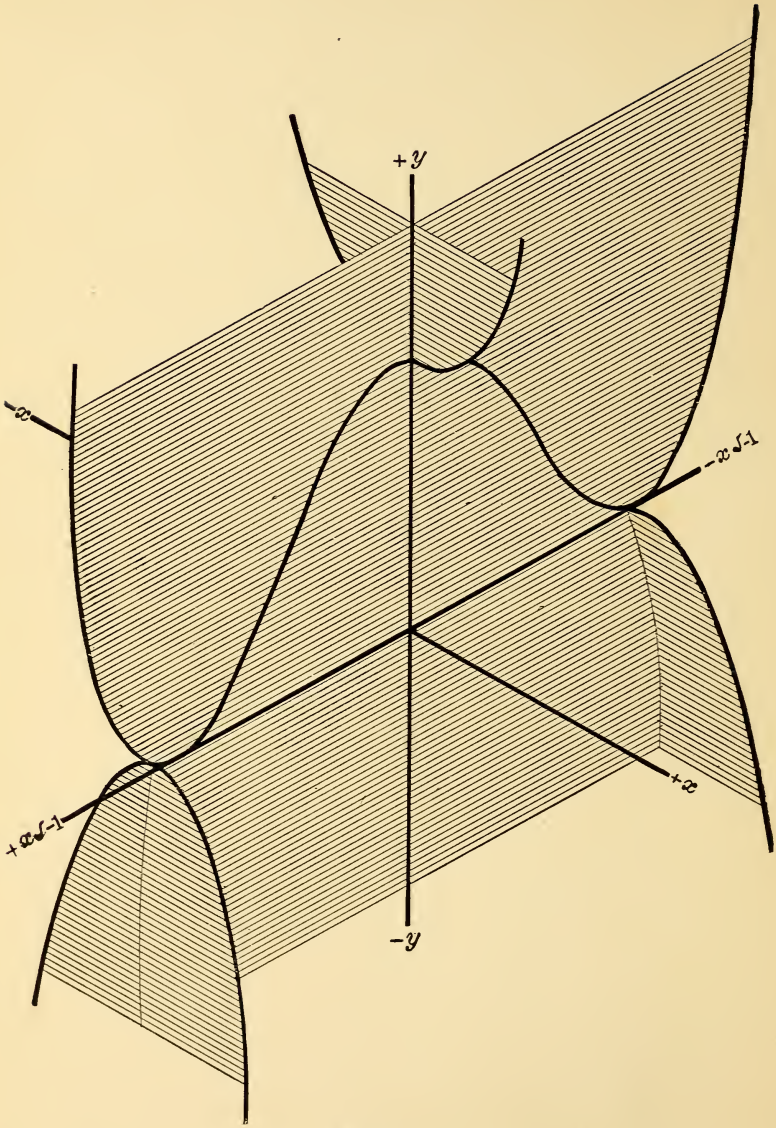

Fig. 59.

revolved about the *y-axis* through one right angle, it will coincide with the curve of the equation

$$y = x^4 - 2x^2 + 1.$$

105. If in either one of these equations we substitute for x, $x\sqrt{-1}$, we shall obtain the other, which is another illustration of the fact that multiplying a quantity by $\sqrt{-1}$ turns it through a right angle. Here, since every value of x is multiplied by $\sqrt{-1}$, the whole curve is turned about the *y-axis* through a right angle.

106. Fig. 60 is the plot of the equation

$$y = x^4.$$

Here the branches are all symmetrical, for we may put $-x$, $x\sqrt{-1}$, or $-x\sqrt{-1}$ in place of x without changing the equation.

107. *To draw the plots from which the roots of any equation of the fourth degree may be derived.* If we substitute for x in the equation

$$y = x^4 \pm ax^2 \pm bx$$

$x = \sqrt{a}x$, and then divide the resulting equation through by a^2 we shall simply change the scale of the curve (Art. 85), and all the equations of the fourth degree may be classified in the three following groups, when y' stands for $\frac{y}{a^2}$ and $\varepsilon = \frac{b\sqrt{a}}{a^2}$:

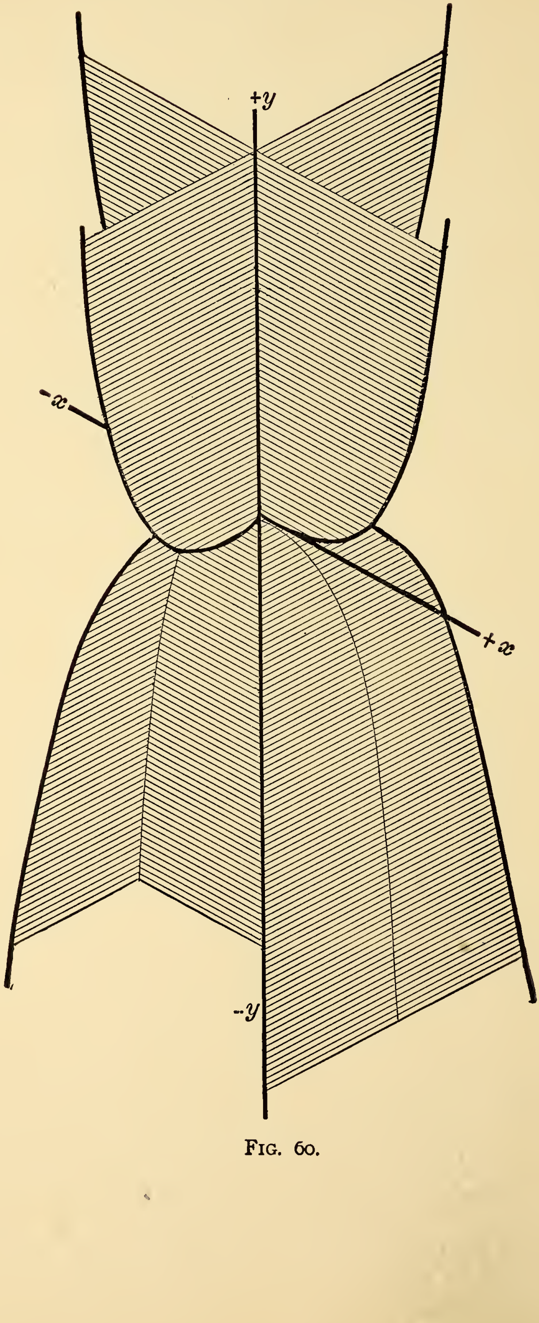

FIG. 60.

(1)
$$\begin{cases} y' = x^4 - x^2; \\ y' = x^4 - x^2 + \varepsilon x; \\ y' = x^4 - x^2 - \varepsilon x. \end{cases}$$

(2)
$$\begin{cases} y' = x^4 + x^2; \\ y' = x^4 + x^2 + \varepsilon x; \\ y' = x^4 + x^2 - \varepsilon x. \end{cases}$$

(3)
$$\begin{cases} y' = x^4; \\ y' = x^4 + \varepsilon x; \\ y' = x^4 - \varepsilon x. \end{cases}$$

108. The first equation in each of these three groups gives a symmetrical curve, and may be plotted according to the method given in the last article.

109. After having drawn the plot of the first equation in any one of these groups, we may plot the *real* part of the second equation of that group as follows :

Plot the straight line $y = \varepsilon x$, and upon the *x-axis* of this plot mark the points corresponding to the values of x assumed for the plot of the first equation of the group. Through these points draw lines parallel to the *y-axis*. Upon each one of these parallels lay off from the line $y = \varepsilon x$ the value of y, taken from the plot of the first equation of the group, which corresponds to the value of x through which this parallel passes. The line drawn through the points so determined will be the curve required.

The plot of the real part of the curve of the third equation

in each group may be drawn in the same way upon the plot of the line

$$y = -\varepsilon x.$$

110. The table of values for plotting the real and imaginary portions of the curves of the second and third equations of each group may be derived from formulas obtained as follows:

Since the second term of the second member of the equation $y = x^4 \pm x^2 \pm \varepsilon x$ is wanting, the sum of the positive values of x must be equal to the sum of the negative values for each value of y; hence the form of the four values of x will be

$$(A) \qquad \begin{cases} x = m + p\sqrt{-1}; \\ x = m - p\sqrt{-1}; \\ x = -m + q\sqrt{-1}; \\ x = -m - q\sqrt{-1}; \end{cases}$$

and forming the factors corresponding to these values of x, as in Art. 102, and reducing, we have

$$-(m^2+p^2)(m^2+q^2) = x^4 + (p^2+q^2-2m^2)x^2 + 2m(p^2-q^2)x,$$

which corresponds term by term with

$$y = x^4 \pm x^2 \pm \varepsilon x;$$

whence

$$p^2 + q^2 - 2m^2 = \pm 1,$$

$$2m(p^2 - q^2) = \pm \varepsilon.$$

Assuming in succession values of m, p and q may be found from the above relations, and y may be found from

$$y = -(m^2 + p^2)(m^2 + q^2).$$

The values of x corresponding may be obtained from group (A). This method will apply whether the values of x are all imaginary, two imaginary and two real, or all real. For when all the values of x are imaginary, p and q will be real;

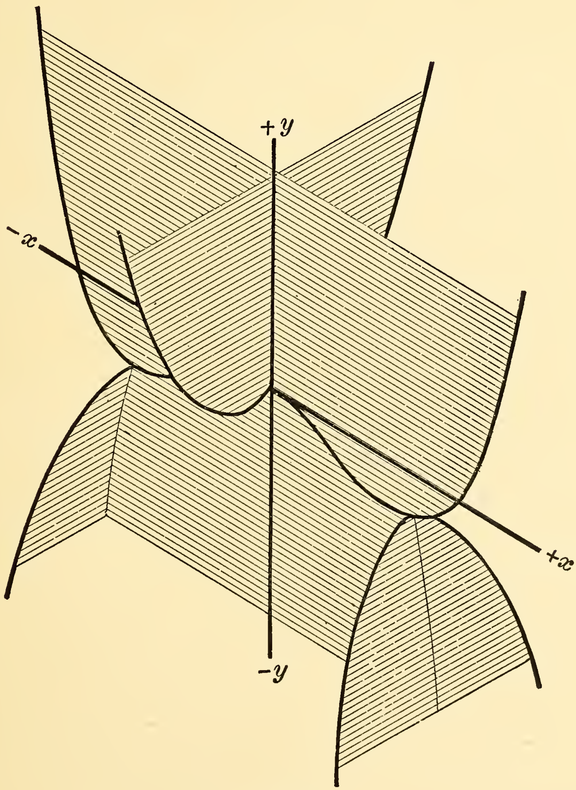

FIG. 61.

when there is one pair of imaginary, either p or q will be real; and when all the values of x are real, p and q will be imaginary.

The above is the plot of the equation

$$y = x^4 - x^2.$$

The following is the plot of the equation

$$y = x^4 - x^2 - \tfrac{1}{10}x.$$

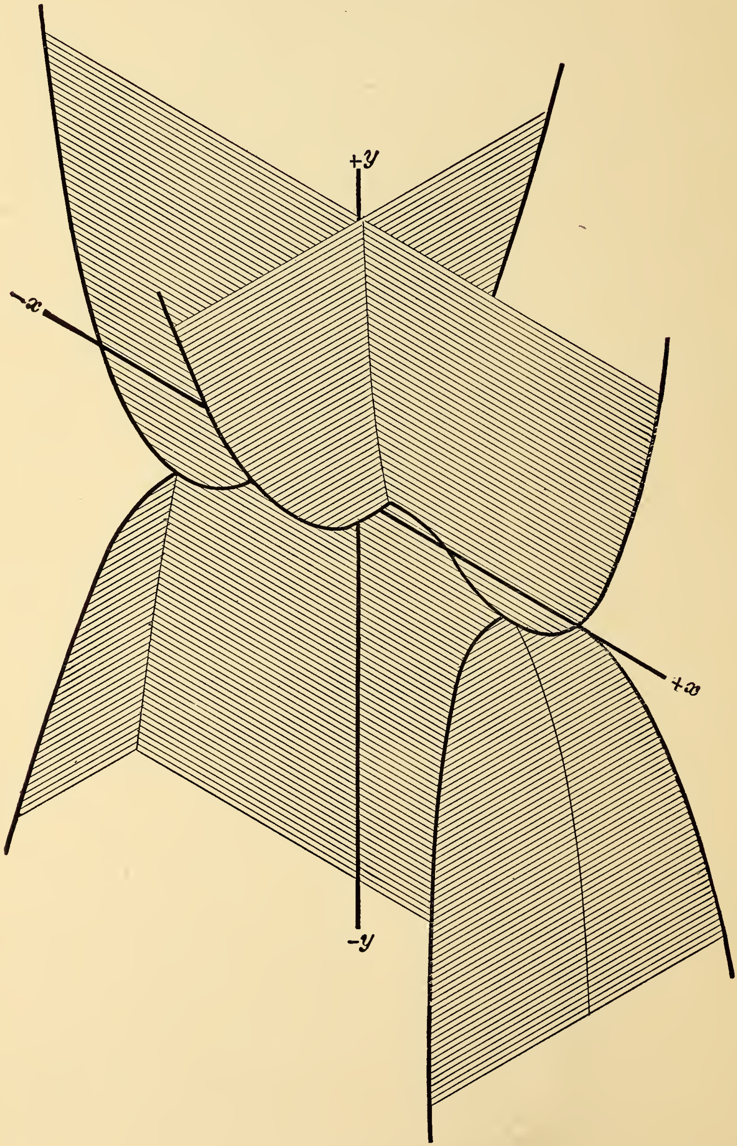

FIG. 62.

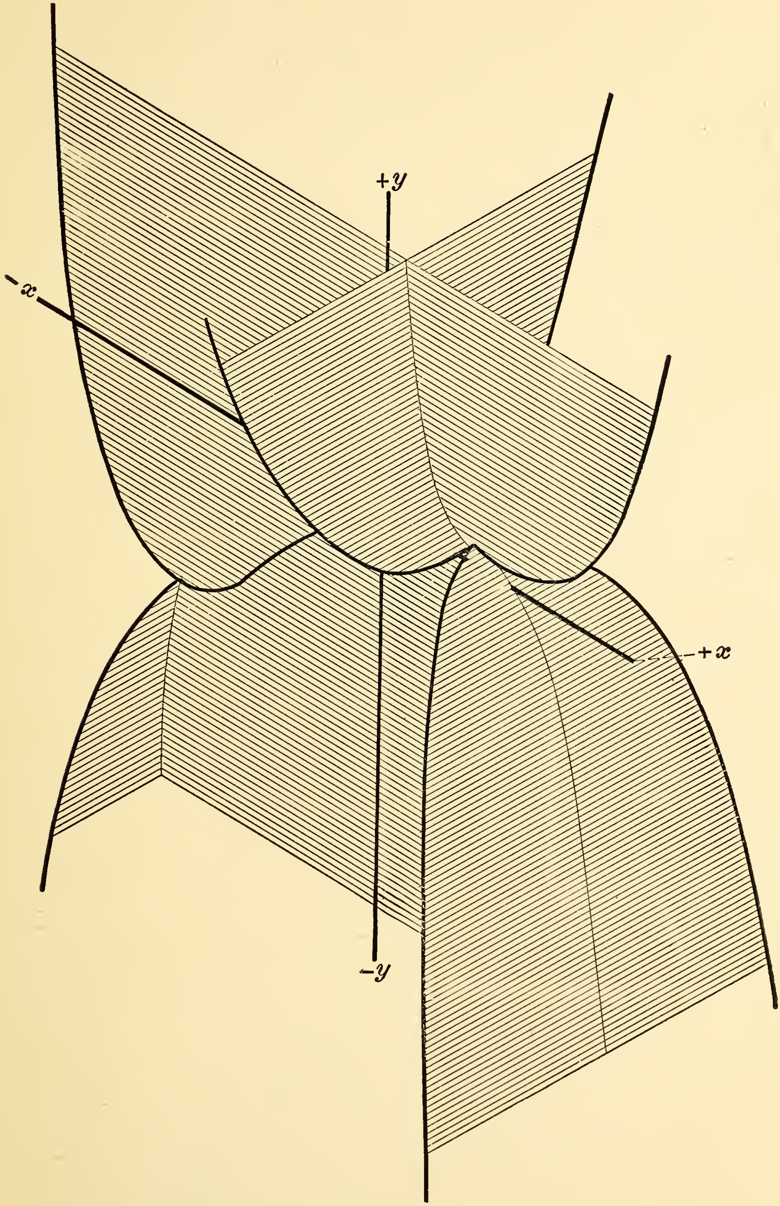

FIG. 63.

This plot is given to show how a small x term affects the form of the real and imaginary branches of the curve.

III. Fig. 63 is the plot of the equation

$$y = x^4 - x^2 + 2\frac{\sqrt{6}}{9}x.$$

By adding an absolute term $-\frac{1}{12}$ to the right-hand member of this equation, we have the case of three equal roots in an equation of the fourth degree. We may derive the conditions of equal roots as follows: The derivative of the equation $x^4 - x^2 + \varepsilon x + q = 0$ is $4x^3 - 2x + \varepsilon$. Find the greatest common divisor of these two expressions.

$$\begin{array}{l}
4x^3-2x+\varepsilon \,)\, 4x^4-4x^2+4\varepsilon x+4q \,\underline{|\, x\quad}\\
\qquad\qquad\quad \underline{4x^4-2x^2+\varepsilon x\qquad\quad}\\
\qquad\qquad\quad +2x^2-3\varepsilon x-4q \,)\, 4x^3-2x+\varepsilon \,\underline{|\, 2x+3\varepsilon}\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{4x^3-6\varepsilon x^2-8qx\qquad}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 6\varepsilon x^2+(8q-2)x+\varepsilon\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{6\varepsilon x^2-9\varepsilon^2 x-12\varepsilon q}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad (9\varepsilon^2+8q-2)x+12\varepsilon q+\varepsilon
\end{array}$$

If this remainder is zero, then $2x^2 - 3\varepsilon x - 4q$ is the greatest common divisor. The remainder will be zero when

$$9\varepsilon^2 + 8q - 2 = 0$$

and $$12\varepsilon q + \varepsilon = 0;$$

whence $$q = -\frac{1}{12} \quad \text{and} \quad \varepsilon = \pm\frac{2\sqrt{6}}{9}.$$

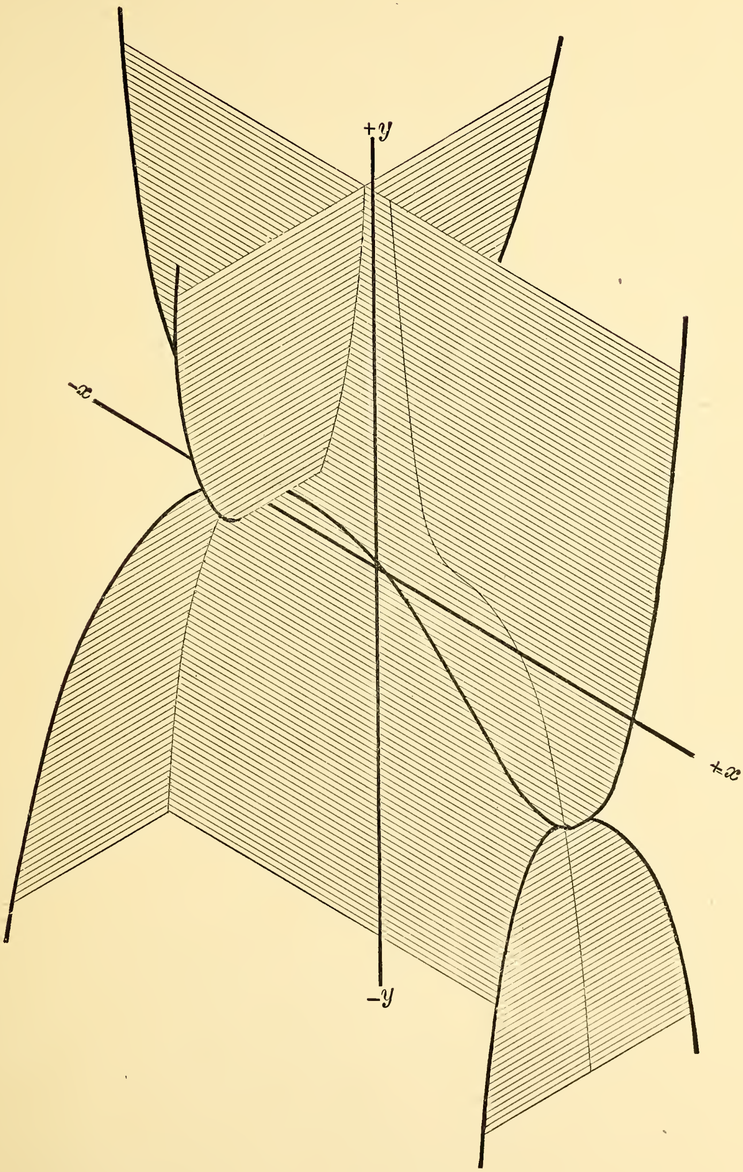

FIG. 64.

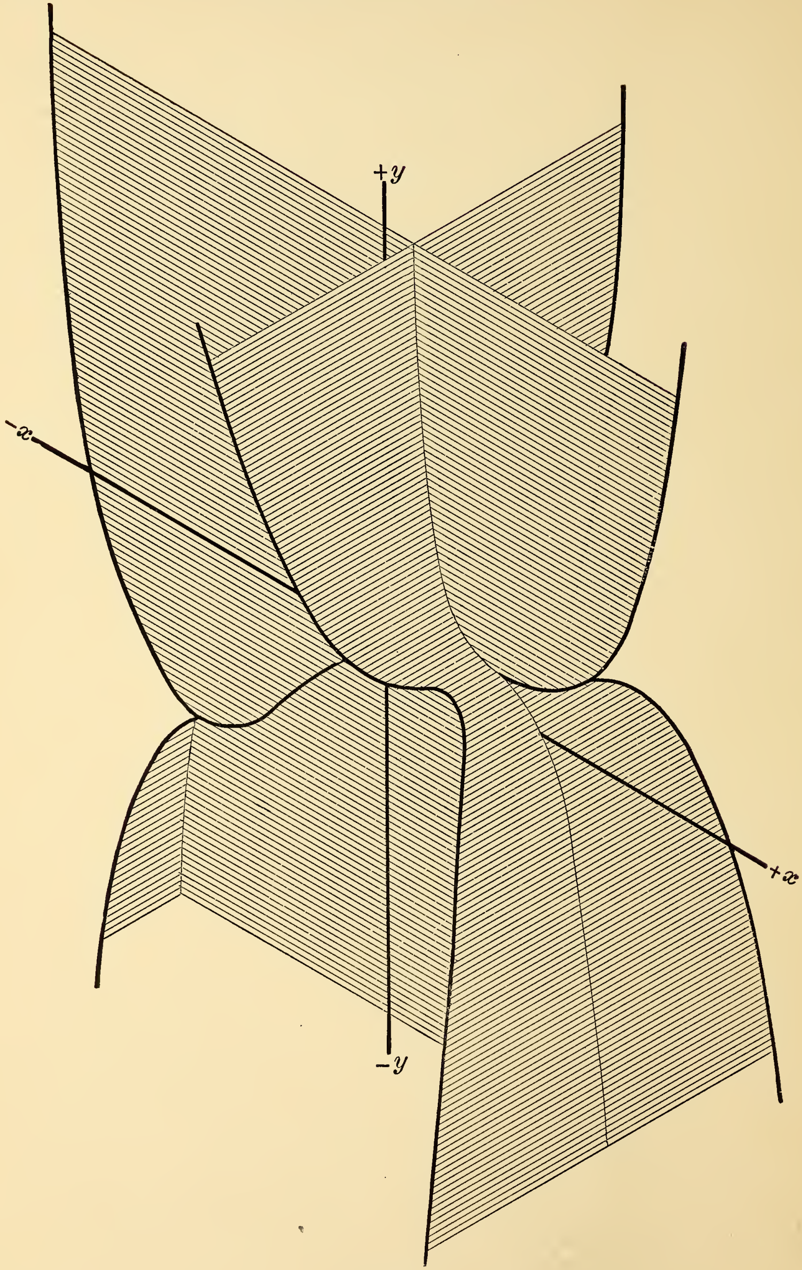

Fig. 65.

Substituting these values in $2x^2 - 3\varepsilon x - 4q$ we find it to be a perfect square. Hence the above are conditions of equal roots. The plot of the equation

$$y = x^4 - x^2 - \frac{2\sqrt{6}}{9}x$$

would give three equal roots if the origin were raised $\frac{1}{12}$.

112. On the opposite page is the plot of

$$y = x^4 - x^2 + .6x.$$

This is selected to show the form of the real and imaginary branches of the curve for a value of ε a little larger than the one required to give equal roots.

On page 136 is the plot of the equation

$$y = x^4 - x^2 - .6x.$$

The curve of this equation is symmetrical to the curve of the equation

$$y = x^4 - x^2 + .6x$$

with respect to the *y-axis*, for one equation may be derived from the other by putting $-x$ in place of x.

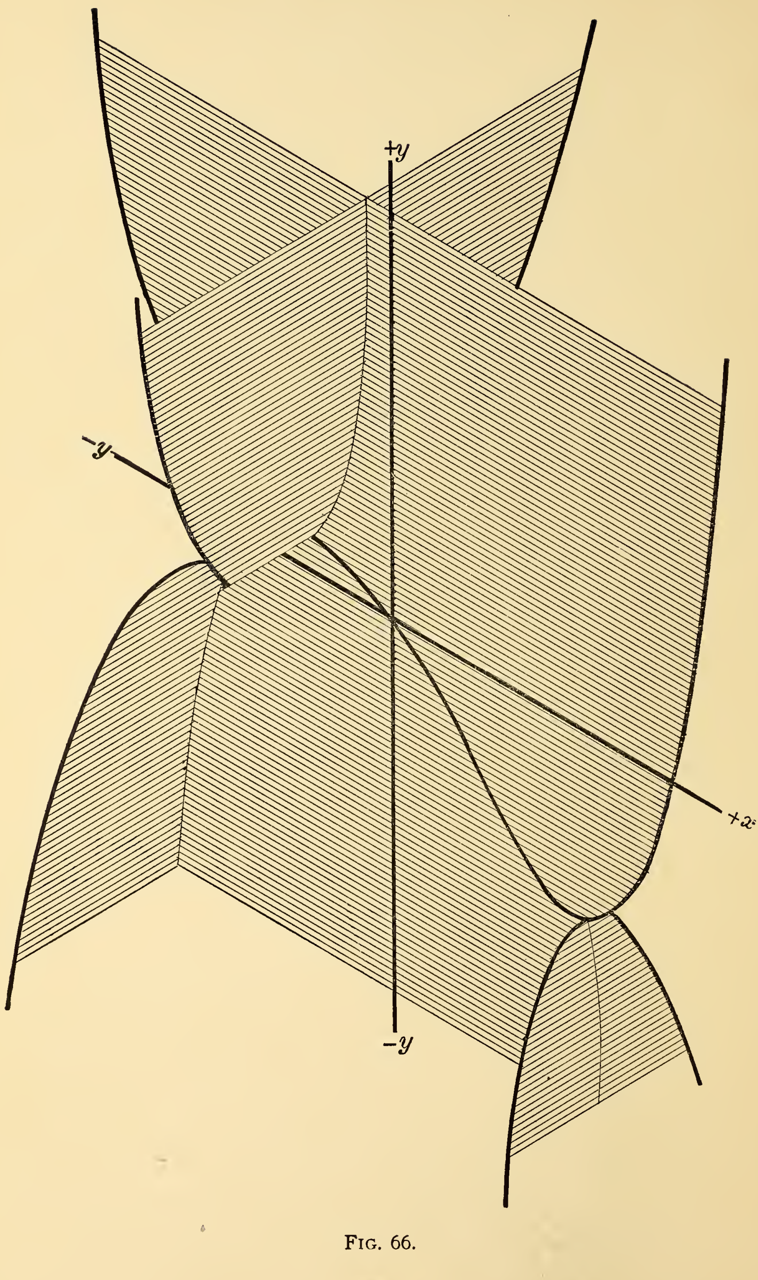

Fig. 66.

113. The plot of the equation

$$y = x^4 + x^2.$$

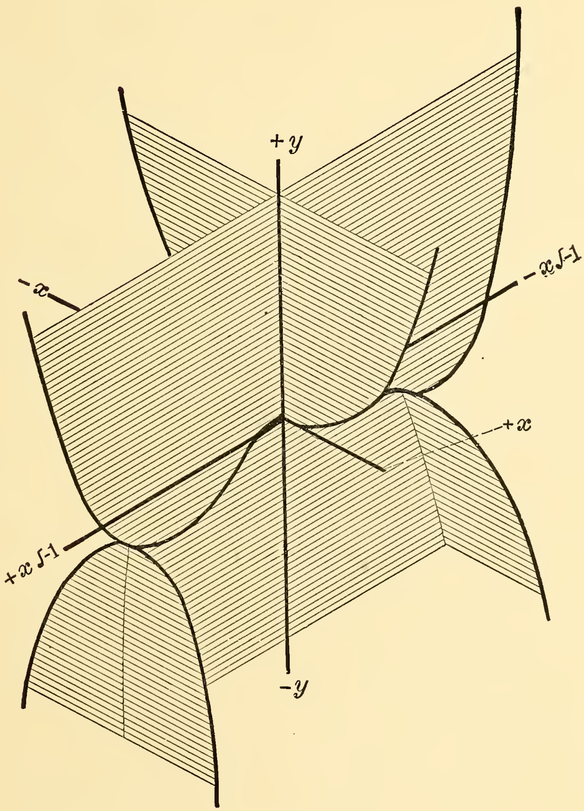

FIG. 67.

It will be seen that the same relation exists between this curve and the curve of the equation $y = x^4 - x^2$ as was found to exist between the curves in Arts. 102 and 103.

The following is the plot of the equation

$$y = x^4 + x^2 + x.$$

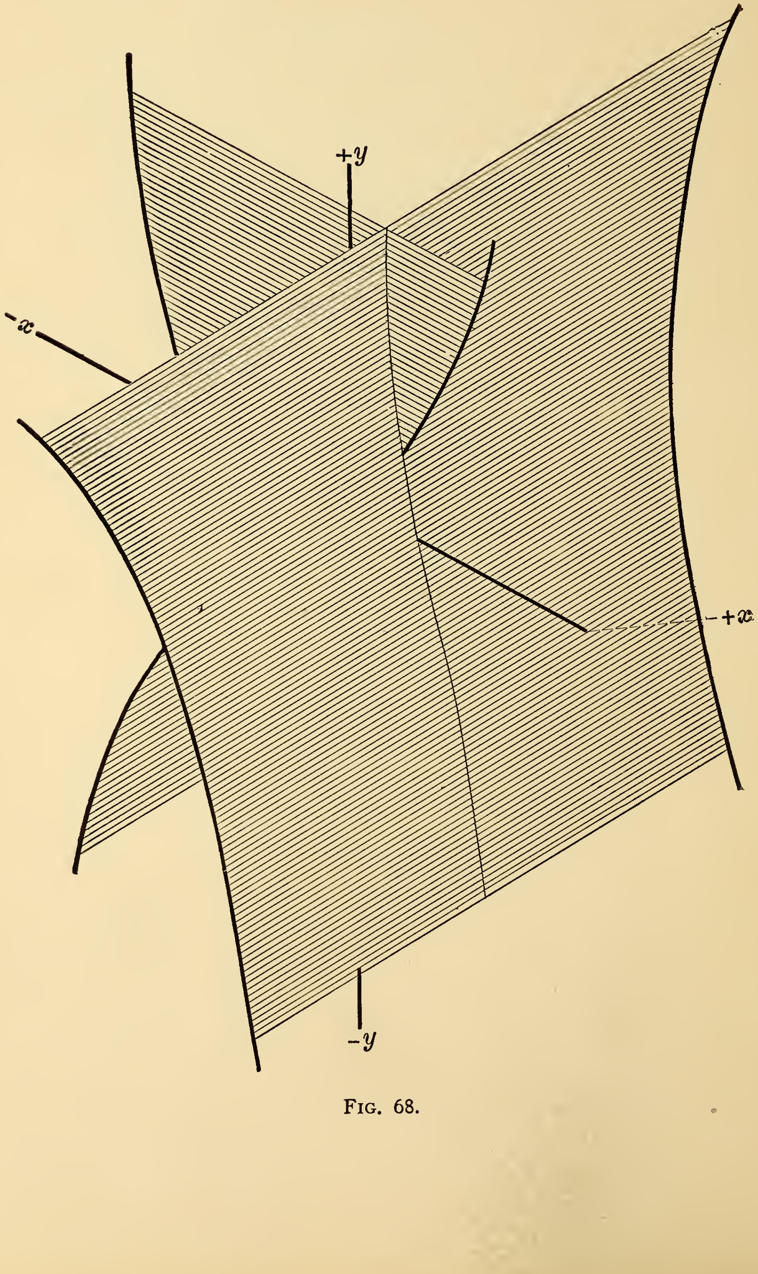

Fig. 68.

In connection with this group we will point out the fact that the coefficient of x may be taken imaginary, and thus give all the forms of the curve of

$$y = x^4 - x^2 \pm \varepsilon x$$

turned through one right angle. To effect this we substitute $x\sqrt{-1}$ for x.

Fig. 69 is the plot of the equation

$$y = x^4 + x^2 - x$$

—an example of the third equation of the second group, where $\varepsilon = 1$.

114. The plot of the equation

$$y = x^4$$

has been given on page 126. This forms the basis of the curves in the third group of page 127.

Fig. 70 is the plot of the equation

$$y = x^4 + x.$$

We have given but one example of the second and third forms of groups (2) and (3), viz. when $\varepsilon = 1$, because that sufficiently illlustrates the general type of all the equations in these forms so long as the coefficients of the powers of x are real numbers.

Fig. 71 is the plot of the equation

$$y = x^4 - x.$$

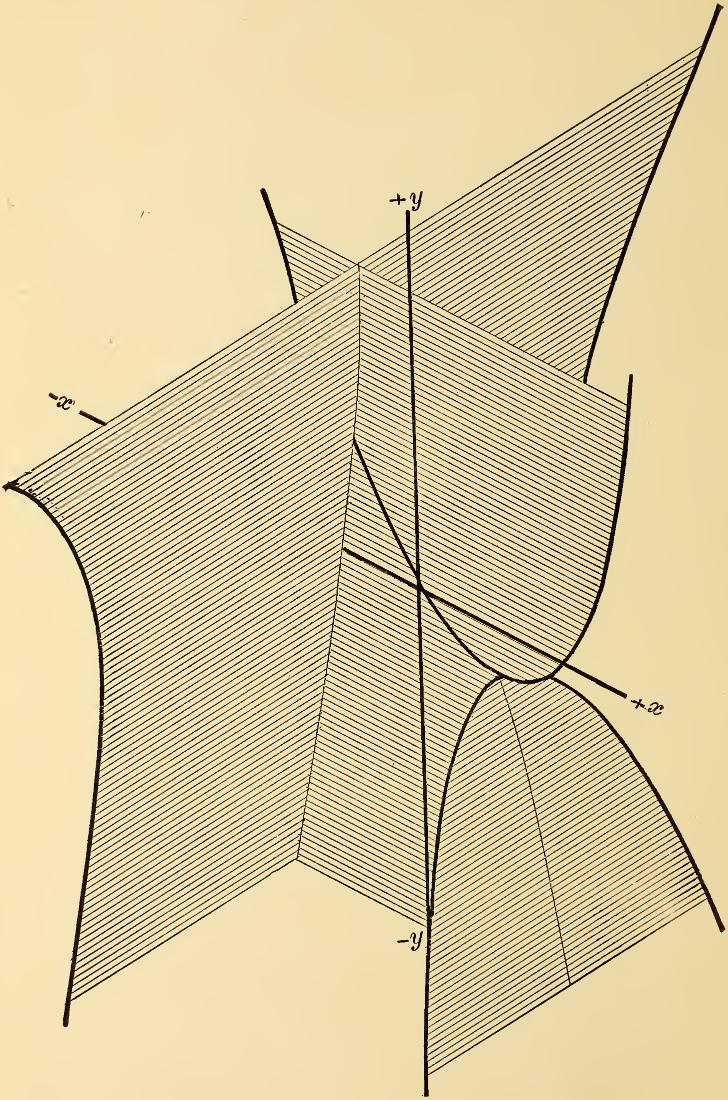

FIG. 69.

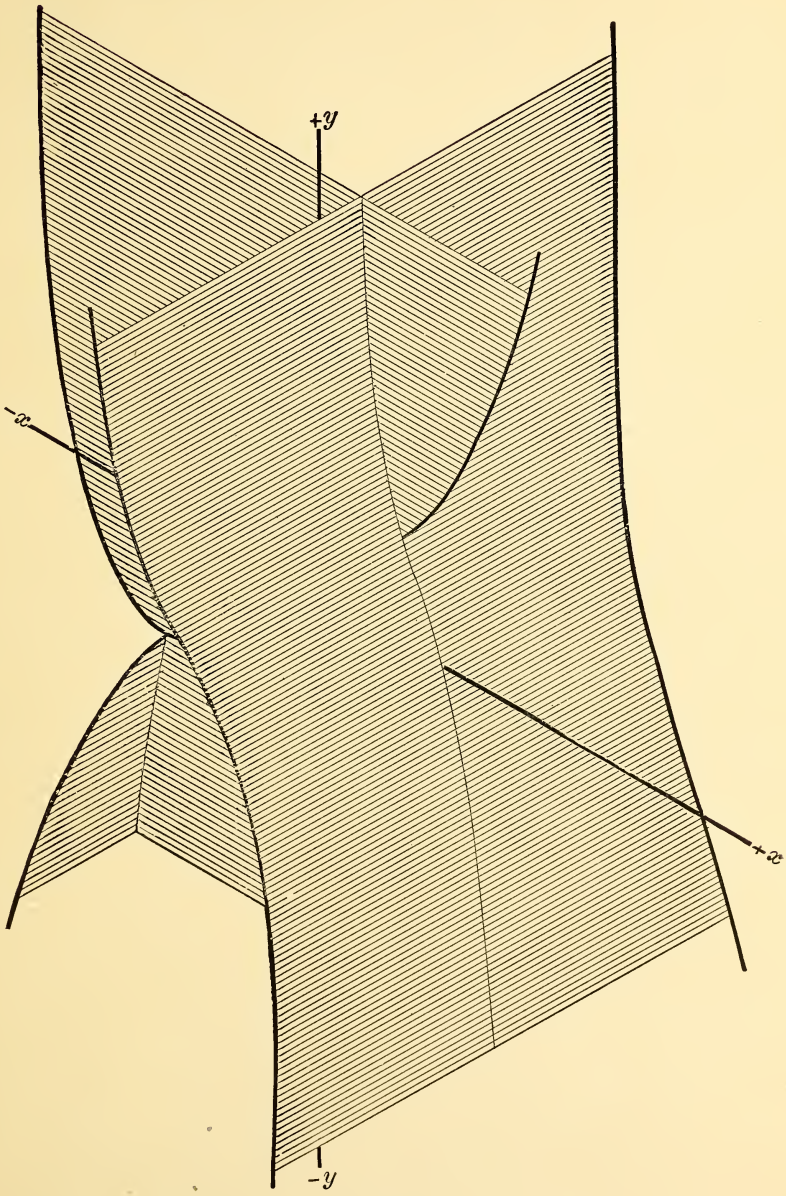

FIG. 70.

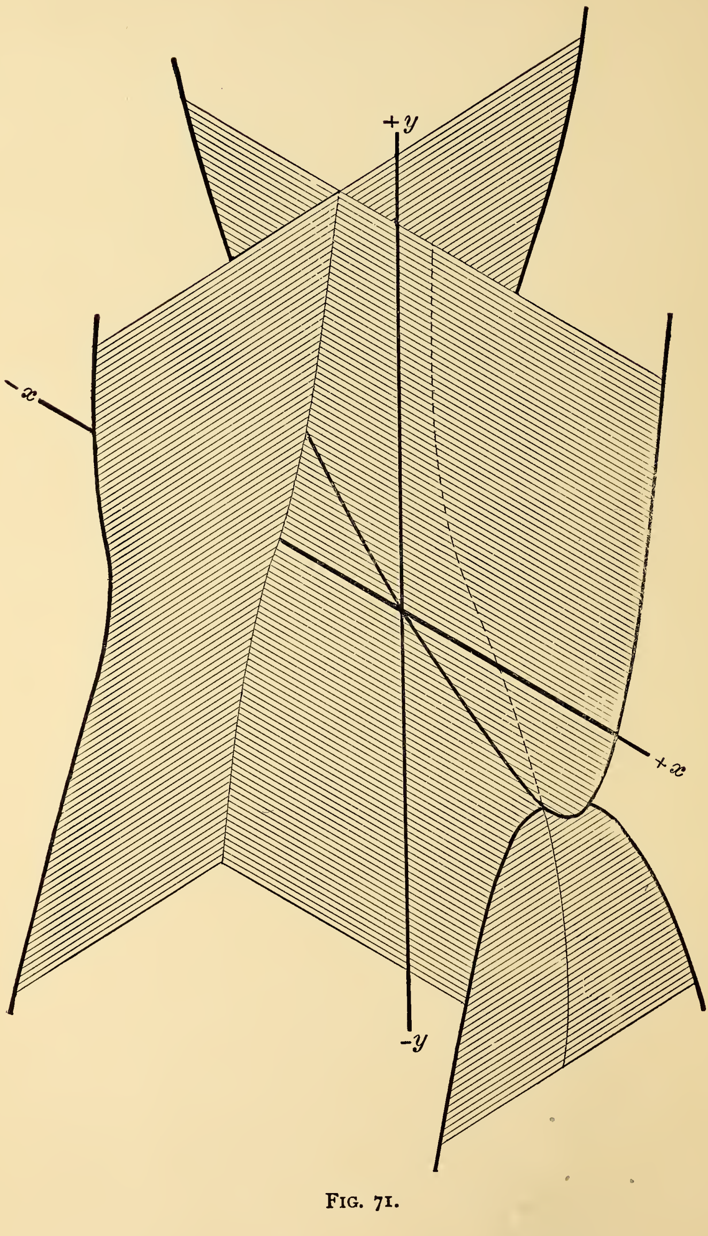

FIG. 71.

115. *Approximation to the Imaginary Roots of Equations of the Fourth Degree.* The real roots of the equation

$$x^4 + bx^2 + cx + d = 0$$

may be found to any degree of accuracy by Double Position or Horner's Method. If there are only two imaginary roots, we may find them by depressing the equation to a quadratic after finding the real roots, and then solving this quadratic.

116. When *all* the roots of the equation are imaginary we may find them by the method of Double Position applied as follows:

The values of x in the equation will be of the form

$$x = m \pm p\sqrt{-1} \quad \text{and} \quad x = -m \pm q\sqrt{-1},$$

and the equation formed from these roots will be

$$x^4 + (p^2 + q^2 - 2m^2)x^2 + 2m(p^2 - q^2)x + (m^2 + p^2)(m^2 + q^2) = 0.$$

Comparing this term by term with

$$x^4 + bx^2 + cx + d = 0,$$

we have

$$p^2 + q^2 - 2m^2 = b;$$

$$2m(p^2 - q^2) = c;$$

$$(m^2 + p^2)(m^2 + q^2) = d.$$

If we assume two values of m in the neighborhood of the correct value, and compute p^2 and q^2 and also d, we shall ob-

tain, by subtracting the values of d in the equation from the computed values of d, *errors* which we can compare with the assumed numbers as in the method of Double Position. In this way we may compute to any degree of accuracy the value of m, and from this p and q.

Examples. Find the roots of the following equations:

$$x^4 - 3x^2 + 2x = -4.$$

$$x^4 + 4x^3 - 2x^2 - \tfrac{1}{10}x = -8.$$

$$x^4 + 3x = 4.$$

$$x^4 + 4x^2 - x = 2.$$

The above equations are to be plotted and the first approximations to the roots obtained from the plots. Then the approximation may be carried on as above so far as desired.

Roots of the Equation $x^n = a$.

117. The equation $x^n = a$ must have n roots real or imaginary, one of which is $x = \sqrt[n]{a}$. We may determine graphically the remaining roots of the equation as follows: Assume any point as an origin, and through this point draw the axes of x and $x\sqrt{-1}$ at right angles. With the origin as a centre and a radius equal to $\sqrt[n]{a}$ describe a circumference. Divide this circumference into n equal parts, beginning the

division with the point where the *x-axis* cuts the circle on the right-hand side of the origin if a is positive. From the several points of division drop perpendiculars to the *x-axis*. The distances from the centre to each of these points of division, measured along the *x-axis* and these several perpendiculars, will be the roots of the equation. The following examples will illustrate the method:

Take the equation $$x^3 = 1.$$

We have by transposing $$x^3 - 1 = 0.$$

Factoring, $$(x - 1)(x^2 + x + 1) = 0.$$

Putting each factor equal to zero and solving, we get

$$x = 1, \quad x = -\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1}, \quad \text{and} \quad x = -\frac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{-1}.$$

Now take the point O as an origin and draw the *x-axis* and the $x\sqrt{-1}$ axis, and then describe a circle with unity for radius and O as a centre. Divide this circumference into three equal parts, beginning at A. Draw the perpendiculars BM and CM. Join DB and DC, also OB and OC.

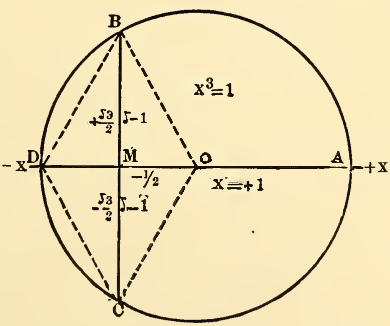

FIG. 72.

The roots of the equation are respectively

$$x = OA = 1,$$

$$x = OM + MB = -\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1},$$

and $$x = OM + MC = -\frac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{-1}.$$

That $OM + MB = -\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1}$ may be proved as follows: Since the circumference is divided into three equal parts, the angle $AOB = 120°$ and $DOB = 60°$. Hence the triangle BDO is equilateral. The perpendicular BM bisects OD at M, and $OM = \frac{1}{2}$. MB the perpendicular of the right-angled triangle $OMB = \sqrt{\overline{OB}^2 - \overline{OM}^2} = \frac{\sqrt{3}}{2}$. Now if we give to these values the signs which they would have from their positions on the plot, OM will be minus, and MB, which is parallel to the $x\sqrt{-1}$ axis, will have $\sqrt{-1}$ attached to it. In the same way

$$OM + MC = -\frac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{-1}.$$

That is, in this example we find the values of x are equal to the distances from the centre of the circle whose radius is unity to each of the points of equal division on the circumference. In the equation $x^3 = a$ the same would hold if the

circle was taken with radius equal to $\sqrt[3]{a}$. If a were a negative quantity the circle would be divided into three equal parts, but the first point of division would not be on the right of the origin on the *x-axis*. In general, the values of x found by solving the equation, when laid off according to the usual interpretation of the signs $+$, $-$, and $\sqrt{-1}$ on the circle whose radius is $\sqrt[n]{a}$, will determine points on the circle, which divide it into n equal parts.

118. If we should write y in place of a in the equation $x^n = a$, the series of values assumed for a, with the corresponding values of x, would give the table of values for plotting the real and imaginary branches of the curve of the equation

$$y = x^n.$$

That is, if we plot the real branch of the equation

$$y = x^n,$$

where the *y-axis* is a vertical line in space, and then revolve this curve so plotted about the *y-axis*, the n positions which this curve takes, viz., when it has completed $\frac{1}{n}$, $\frac{2}{n}$, $\frac{3}{n}$, etc., of one revolution, will be the positions of the real and imaginary branches of the curve. Where n is even and a or y is negative, the branches below the *x-axis* or rather below the horizontal plane passing through the *x-axis*, will all be of the form of the real branch of the curve inverted. Now if planes

be passed through the *y-axis* bisecting the angular spaces between the branches above the horizontal plane just mentioned, and these planes be continued below this horizontal plane, then, when the inverted curve in the course of its revolution about the *y-axis* comes into coincidence with these vertical planes, it will coincide with the imaginary branches of the curve which are situated below the horizontal plane.

119. To find the roots, therefore, of any equation $x^n = a$ on a model constructed as above of

$$y = x^n,$$

measure off a distance a on the *y-axis* and pass through this point a plane perpendicular to the *y-axis*. The points where this plane cuts the branches of the curve will lie on a circle of radius $\sqrt[n]{a}$, and upon this the roots of $x^n = a$ may be found by measurement. The same may be found by measurements on the plots of these equations drawn in isometric projection.

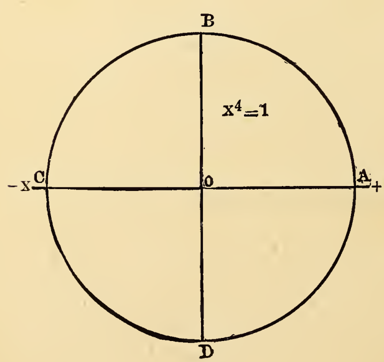

FIG. 73.

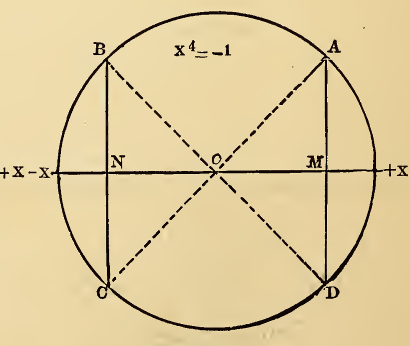

FIG. 74.

Fig. 81 is such a plot of $y = x^3$.

Fig. 82 of $y = x^4$.

Fig. 83 of $y = x^5$.

Fig. 84 of $y = x^6$.

The roots of $x^4 = 1$ are

$$+1, \quad -1, \quad +\sqrt{-1}, \quad -\sqrt{-1},$$

or, Fig. 73, OA, OC, OB, and OD.

The roots of $x^4 = -1$ are

$$+\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}\sqrt{-1} \quad \text{and} \quad -\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}\sqrt{-1},$$

or, Fig. 74,

$OM + MA$, $OM + MD$, $ON + NB$, and $ON + NC$.

The five roots of $x^5 = 1$ may be found as follows:

Dividing $x^5 - 1 = 0$ by $x - 1$, since 1 is a root of the equation, gives

$$x^4 + x^3 + x^2 + x + 1 = 0;$$

and dividing by x^2,

$$x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = 0.$$

Put $v = x + \frac{1}{x}$, then $v^2 = x^2 + 2 + \frac{1}{x^2}$, whence

$$v^2 + v = 1 \quad \text{and} \quad v = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5},$$

and the values of x by substitution become—

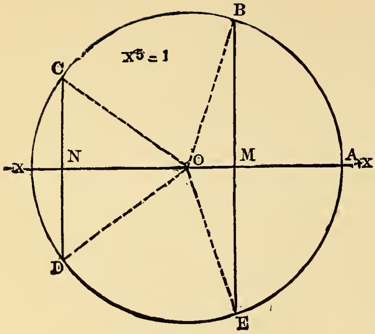

FIG. 75.

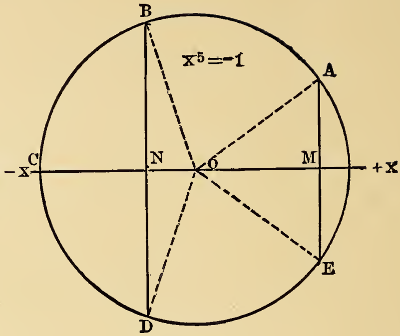

FIG. 76.

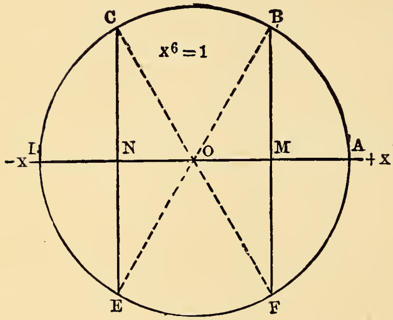

FIG. 77.

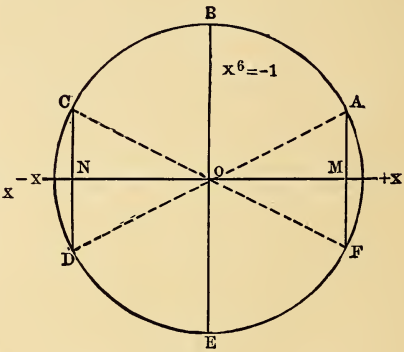

FIG. 78.

1, $\qquad \frac{1}{4}[\sqrt{5} - 1 \pm \sqrt{-10 - 2\sqrt{5}}]$

and $\qquad \frac{1}{4}[-\sqrt{5} - 1 \pm \sqrt{-10 + 2\sqrt{5}},]$

and if the circle be divided into five equal parts it may be shown from the properties of the pentagon that these numbers are the values of the lines which give the measurements of the roots of the equation Fig. 75. The roots of $x^6 = -1$

are given on Fig. 76. Figs. 77 and 78 give the roots of $x^6 = +1$ and $x^6 = -1$. Figs. 79 and 80 show these roots on the figures drawn in isometric projection. Fig. 79 is an outline of the upper part of Fig. 84, and Fig. 80 is the outline of

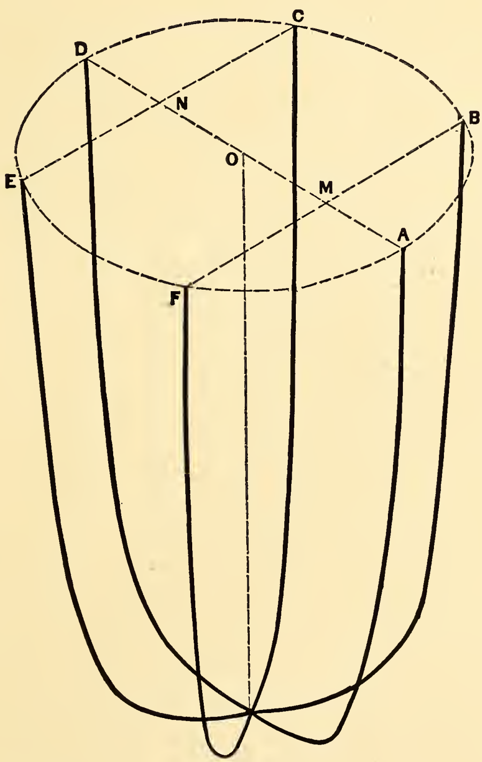

FIG. 79.

the lower part. These figures are drawn on a scale such that values of x are one third the size of the same values of y.

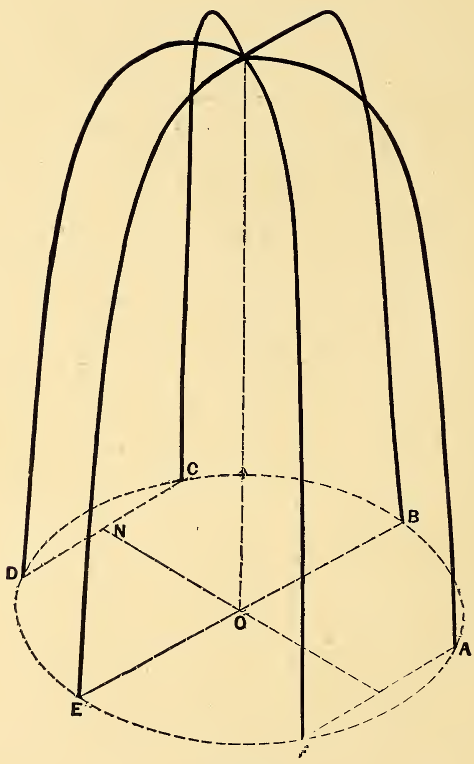

FIG. 80.

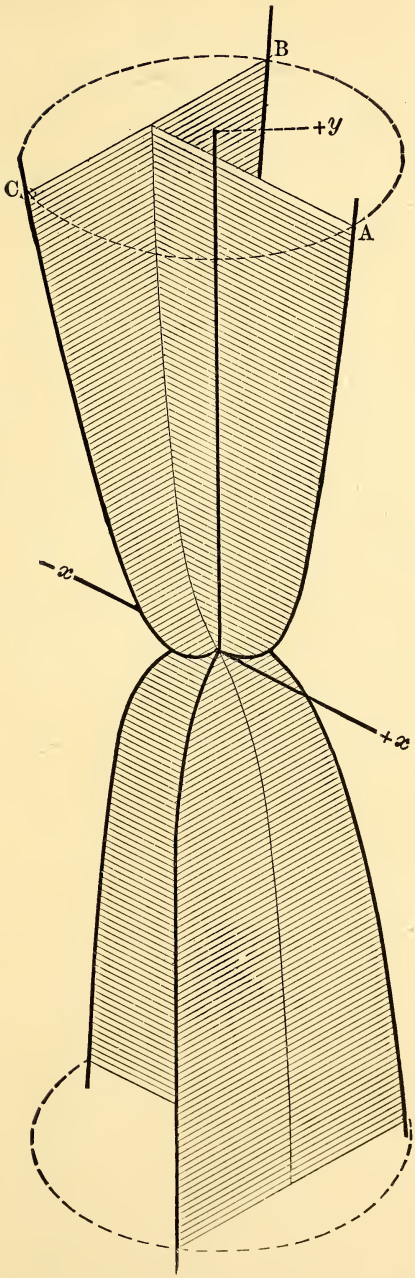

FIG. 81.

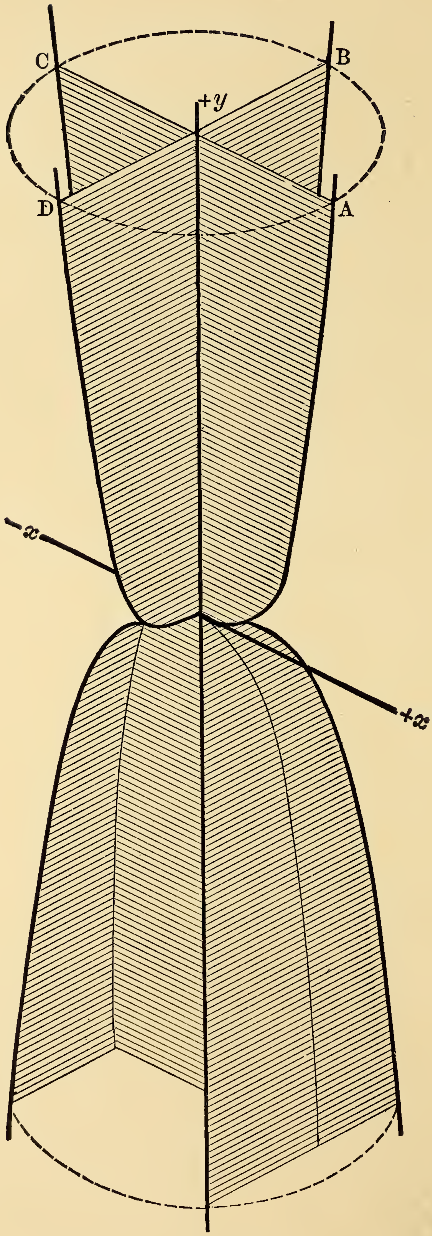

FIG. 82.

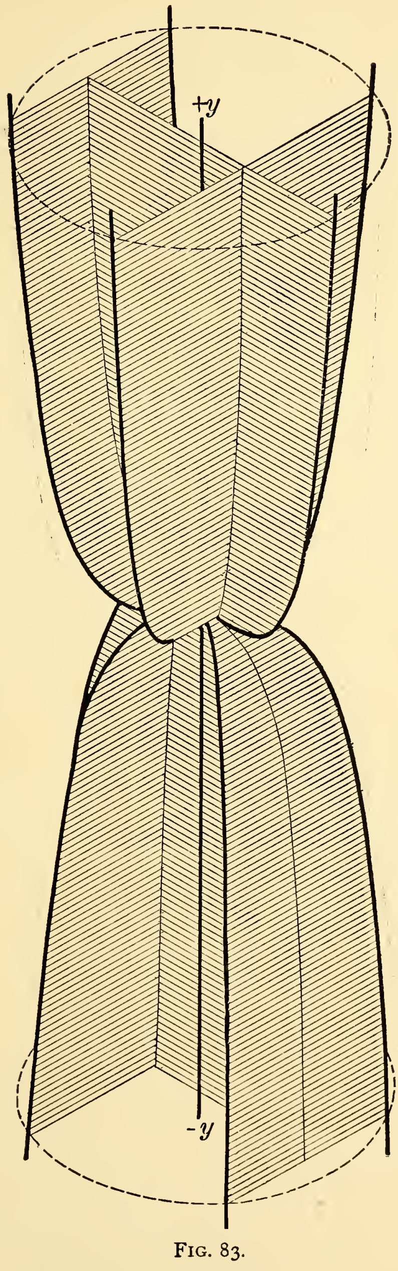

FIG. 83.

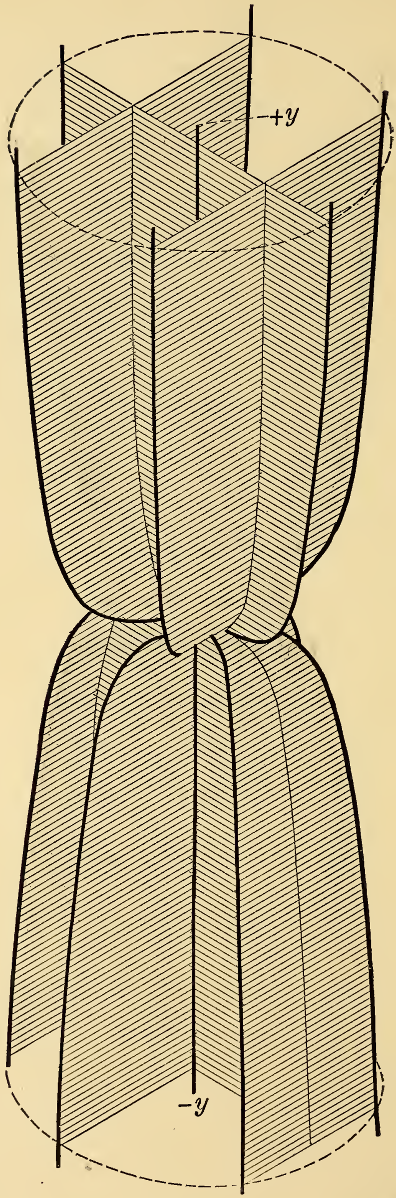

Fig. 84.

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